RIGIDITY FOR INFINITELY RENORMALIZABLE AREA-PRESERVING MAPS

D. GAIDASHEV, T. JOHNSON , M. MARTENS

ABSTRACT. The period doubling Cantor sets of strongly dissipative Hénon-like maps with different average Jacobian are not smoothly conjugated. The Jacobian Rigidity Conjecture says that the period doubling Cantor sets of two-dimensional Hénon-like maps with the same average Jacobian are smoothly conjugated. This conjecture is true for average Jacobian zero, e.g. the one-dimensional case. The other extreme case is when the maps preserve area, e.g. the average Jacobian is one. Indeed, the period doubling Cantor set of area-preserving maps in the universality class of the Eckmann-Koch-Wittwer renormalization fixed point are smoothly conjugated.

CONTENTS

1. INTRODUCTION

The simplest dynamics beyond periodic behavior can be found in systems at the accumulation of period doubling. These systems have a period doubling Cantor set whose dynamics can be very well understood in terms of the periodic orbits which accumulate on this set. The consecutive approximating periodic orbits are related by so-called period doubling bifurcations. This simplest non-period behavior is often observed in models and in nature. In particular, the transition to chaos in strongly dissipative systems often occurs with such a period doubling Cantor set.

The topological properties of these period doubling Cantor sets are simple and very well understood. In the mid-1970's Feigenbaum [F1, F2] and, independently, Coullet and Tresser [CT, TC], discovered universal geometrical properties of these Cantor sets at the transition to chaotic behavior in one-dimensional dynamics. Coullet and Tresser conjectured that the universal geometry at transition to chaos in one-dimensional dynamics will also be observed in higher dimensional systems. This conjecture has been confirmed by many numerical and physical experiments.[1]

Feigenbaum-Coullet-Tresser introduced renormalization in dynamics to explain the observed geometrical universality. The renormalization operator acts as a microscope. By repeatedly applying, one can describe the small scale geometrical structure by renormalizations, e.g. systems associated to the smaller scales. The corresponding renormalization operator has a unique hyperbolic fixed point. The dynamics of the renormalization fixed point, which is itself a one-dimensional system, and the behavior of the renormalization operator around this fixed point determine the asymptotic small scale properties. This explains the observed universality.

The renormalization technique has been generalized to many types of dynamics. However, a rigorous study of universality has been surprisingly difficult and technically sophisticated and so far has only been thoroughly carried out in the case of one-dimensional maps, on the interval or the circle, see [AL, FMP, He, L, Ma, McM, MS, S, VSK, Y]. There is, however, at present no deep understanding of universality in conservative systems, other than in the case of the universality for systems "near integrability" [AK, AKW, EKW2, K1, K2, K3, G1, Ko, Kh].

Convergence of renormalization implies universal geometry at asymptotic small scale around a certain point. A stronger property is rigidity: there is universal geometry at asymptotic small scale around all points. All period doubling Cantor sets are topologically equivalent. Between two of them there will be a homeomorphism conjugating the dynamics. A priori there is no reason to belief that these homeomorphisms have some smoothness. Rigidity means that there are conjugating homeomorphism which are smooth. The topology determines the small scale geometry.

Indeed, the period doubling Cantor sets in one-dimensional dynamics are rigid. This is surprising. Especially because the Cantor sets have

---

[1]This conjecture should be taken with caution as not every transition to chaos is related to accumulation of period doubling.

very rich geometry: there are essentially no two points with the same asymptotic small scale geometry, see [BMT]. Many numerical and physical experiments show that exactly the same universal geometry from one-dimensional dynamics occurs also in some dissipative higher dimensional systems. Surprisingly, the rigidity phenomenon is more delicate in higher dimensions.

Strongly dissipative two-dimensional Hénon-like maps can be thought of as two dimensional perturbations of one-dimensional systems. In [CEK1] and [CLM] two renormalization schemes were developed for strongly dissipative Hénon-like maps at the accumulation of period doubling. This explained the observed one-dimensional universal geometry present in these Hénon-like maps. Surprisingly, the period doubling Hénon-like Cantor sets are not smoothly conjugated to their one-dimensional counter part, see [CLM, LM1].

The average Jacobian, e.g. the average rate of dissipation, plays a role. The Cantor set of systems with different average Jacobian are not smoothly conjugated, see [CLM, LM1]. Nevertheless, the geometry of the one-dimensional period doubling Cantor set is still present. The conjugations between the Cantor sets of strongly dissipative Hénon-like maps is almost everywhere, with respect to the natural measure on the Cantor set, smooth. This phenomenon is called *Probabilistic Rigidity*, see [LM1, LM3]. Small scale geometry has a probabilistic nature in higher dimensions.

As we discussed before, different average Jacobians are an obstruction to rigidity. However, this is not contradicting the rigidity paradigm *topology determines geometry*. Namely, in [LM2] it is shown that maps with different average Jacobian are not topologically equivalent. The average Jacobian is a topological invariant. It describes topological aspects of the heteroclinic web, e.g. the network of stable and unstable manifolds of the period orbits. This leads to the

Jacobian Rigidity Conjecture. *The period doubling Cantor sets of two-dimensional Hénon-like maps with the same average Jacobian are smoothly conjugated.*

This conjecture is true for average Jacobian zero, e.g. the one-dimensional case. The other extreme case when the maps preserve area, e.g. the average Jacobian is one, is described in Theorem 4.1. It says,

Rigidity for Area-preserving Maps. *The period doubling Cantor set of area-preserving maps in the universality class of the Eckman-Koch-Wittwer renormalization fixed point are smoothly conjugated.*

Area-preserving maps at accumulation of period doubling are observed by several authors in the early 80's, see [DP, Hl, BCGG, Bo, CEK2, EKW1]. In [EKW2] Eckmann, Koch and Wittwer introduced a period doubling renormalization scheme for area preserving maps and described the hyperbolic behavior of the renormalization operator in a neighborhood of a renormalization fixed point. In particular, they observed universality for maps at the accumulation of period doubling.

The maps in the universality class of the Eckmann-Koch-Wittwer renormalization fixed point are at the accumulation of period doubling. It was shown in [GJ1] that these maps do have a period doubling Cantor set and the Lyapounov exponents are at most zero. Moreover, for maps in a certain strong stable part of the renormalization fixed point, a space with finite codimension, their period doubling Cantor sets are at least bi-Lipschitz conjugated. In this paper we improve the conclusions of [GJ1]: rigidity holds in the whole stable manifold of the Eckmann-Koch-Wittwer renormalization fixed point. Moreover, the conjugations are at least $C^{1+\alpha}$ with $\alpha = 0.237$.

2. Preliminaries

Given a domain $\mathcal{D} \subset \mathbb{C}^2$, let $D \subset \text{int}(\mathcal{D} \cap \mathbb{R}^2)$ be compactly contained in the real slice. Assume $(0,0) \in D$. An *area-preserving map* $F : D \to F(D) \subset \mathbb{R}^2$ will mean a real symmetric map which has a holomorphic extension to $\mathcal{D}$ and is an exact symplectic diffeomorphism onto its image with the following properties

    1) $F(0,0) = (0,0)$,
    2) $T \circ F \circ T = F^{-1}$, where $T(x,u) = (x,-u)$ (*reversibility*),
    3) $\partial_u x' \neq 0$ with $(x',u') = F(x,u)$ (*twist condition*).

The collection of such maps is denoted by $\text{Cons}(D)$.

Our subject are maps at the accumulation of period doubling in a neighborhood of the Eckmann-Koch-Wittwer renormalization fixed point, [EKW2]. We will need stronger and additional estimates than the spectral information obtained in [EKW2]. These estimates were obtained in [GJ2]. Although both [EKW2] and [GJ2] study a neighborhood of the same fixed point, the used constructions differ. We do need the estimates form [GJ2] and summerize these results here.

In [GJ2] Gaidashev and Johnson construct simply connected domains $D \subset \mathcal{D} \subset \mathbb{C}^2$ and adapt the renormalization scheme from [EKW2]. This renormalization scheme is defined on a neighborhood $\text{Cons}_0(D)$ of $F_* \in \text{Cons}(D)$, where F_* corresponds to the Eckman-Koch-Wittwer fixed point. Each map $F \in \text{Cons}_0(D)$ has a unique periodic orbit of period 2. This orbit has one point $(p_F, 0)$ with $p_F < 0$. The dependance

$$F \mapsto p_F$$

is analytic. Furthermore there are analytic function

$$F \mapsto \lambda_F \in (-\infty, 0)$$

and

$$F \mapsto \mu_F \in (0, \infty).$$

The rescaling which will be used to renormalize is

$$\Lambda_F : (x, u) \mapsto (\lambda_F x + p_F, \mu_F u).$$

The renormalization operator $R : \text{Cons}_0(D) \to \text{Cons}(D)$ is defined by

$$RF = \Lambda_F{}^{-1} \circ F \circ F \circ \Lambda_F.$$

The results from [GJ2] which will be used in the sequel are collected in the following Theorem, Lemma, and Proposition. We will denote the C^2-norm of maps by $||.||_{C^2}$.

Theorem 2.1. *There exists $F_* \in Cons_0(D)$ such that*
1) *$Cons_0(D)$ is a neighborhood of F_*.*
2) *F_* is a hyperbolic fixed point of the renormalization operator.*
3) *F_* has a one-dimensional local unstable manifold.*
4) *F_* has a codimension one local stable manifold $W_{loc}^s(F_*)$.*
5) *F^* has a codimension two local strong stable manifold $W_{loc}^{ss}(F_*) \subset W_{loc}^s(F_*) \subset Cons_0(D)$.*

6) *There exist a distance d on $W_{loc}^{ss}(F_*)$, and $\nu < 0.126$ such that for every $F, \tilde{F} \in W_{loc}^{ss}(F_*)$*

$$||F - \tilde{F}||_{C^2} \le d(F, \tilde{F})$$

and

$$d(RF, R\tilde{F}) \le \nu \cdot d(F, \tilde{F}).$$

In particular,

$$d(R^n F, F_*) \le \nu^n \cdot d(F, F_*).$$

7) *The one dimensional family defined by $F_t = h_t^{-1} \circ F_* \circ h_t$, where $h_t : D \to h_t(D) \subset \mathbb{R}^2$ is the diffeomorphism defined by*

$$h_t(x, u) = (x + tx^2, \frac{u}{1 + 2tx}),$$

with $|t| \le c_0$, is contained in the stable manifold $W_{loc}^s(F_*)$ and is transversal to the strong stable manifold $W_{loc}^{ss}(F_*) \subset W_{loc}^s(F_*)$ and intersects only in F_*.

8) *The map Λ_F depend analytically on F.*

For $F \in \mathrm{Cons}_0(D)$ we will use the notation

$$\psi_0 = \psi_0(F) = \Lambda_F : D \to \mathbb{R}^2$$

and

$$\psi_1 = \psi_1(F) = F \circ \Lambda_F : D \to \mathbb{R}^2$$

In the next sections we will consider infinitely renormalizable maps. These are maps for which all $R^n F$, $n \ge 0$, are defined. For such maps we will consider

$$\Psi_w^n = \psi_{w_1}(F) \circ \psi_{w_2}(RF) \circ \cdots \circ \psi_{w_n}(R^{n-1}F) : \mathrm{Dom}(R^n F) \to \mathrm{Dom}(F),$$

with $w_k \in \{0, 1\}$.

Lemma 2.2. *For $F \in Cons_0(D)$ there are analytically defined simply connected domains $(p_F, 0) \in B_0(F) \subset D$ and $B_1(F) \subset D$ such that*

$$(2.1) \qquad\qquad B_1(F) \cap B_0(F) = \emptyset,$$

and

$$(2.2) \qquad\qquad F^2(B_0(F)) \cap B_0(F) \ne \emptyset.$$

Moreover,

$$\psi_w(B_0(RF) \cup B_1(RF)) \subset B_w(F),$$

with $w \in \{0, 1\}$.

Proposition 2.3. *There exists $0 < \theta_1 < \theta_2 < 1$ such that for $F \in W_{loc}^s(F_*)$ we have*

$$\theta_1^4 \cdot |v| \le |D\Psi_w^4(x, u)v| \le \theta_2^4 \cdot |v|$$

for every $w \in \{0,1\}^4$ and $(x, u) \in B_0(F) \cup B_1(F)$. Moreover,

$$(2.3) \qquad \frac{\theta_2 \nu}{\theta_1} < 1.$$

Remark 2.1. The following estimates are obtained in [GJ2].

$$(2.4) \qquad\qquad \theta_1 \ge 0.061$$

$$(2.5) \qquad\qquad \theta_2 \le 0.249$$

$$(2.6) \qquad\qquad \nu < 0.126$$

$$(2.7) \qquad\qquad \frac{\theta_2 \nu}{\theta_1} < 0.515$$

Remark 2.2. The estimate (2.3) plays a crucial role in the proof of the rigidity Theorem 4.1. The optimal way to describe this condition is in terms of F_* and the derivative of renormalization $DR(F_*)$ at that point. Namely, let

$$\sigma^* = \text{Spectral radius}(DR(F_*)|T^{ss}),$$

where T^{ss} is the tangent space to $W_{loc}^{ss}(F_*)$ at F_*. Morever, let

$$\theta_1^* = \inf_{w \in \{0,1\}^\infty, (x,u) \in D, ||v||=1} \{\liminf_{n \to \infty} ||D\Psi_{w^n}^n(F_*)(x, u)v||^{\frac{1}{n}}\},$$

where w^n is the truncation of the infinite word w to its first n letters. And similarly, let

$$\theta_2^* = \sup_{w \in \{0,1\}^\infty, (x,u) \in D, ||v||=1} \{\limsup_{n \to \infty} ||D\Psi_{w^n}^n(F_*)(x, u)v||^{\frac{1}{n}}\}.$$

The numbers θ_1^* and θ_2^* reflect geometrical properties of the period doubling Cantor set of F_*, see §3. The rigidity Theorem can be proved under the condition

$$\frac{\theta_2^* \sigma^*}{\theta_1^*} < 1.$$

The derivative of renormalization at the fixed point is a compact operator. In particular, rigidity can be proved on a finite codimension subspace where the contraction is strong enough. The numerical estimates from [GJ2] show that only the weakest stable direction is not strong enough. Luckily, this weakest stable direction corresponds to a one-dimensional family of analytically conjugated maps. The authors do not have a conceptual explanation for this coincidence.

The Appendix is used to present some elementary estimates on the error term of affine approximations of diffeomorphisms. In this section we used coordinates (x, u) to denote a point. In the sequel we will not need anymore the coordinates and denote points simply by x, etc.

3. THE CANTOR SET

In this section we construct the invariant Cantor set for infinitely renormalizable maps. This cantor set is intrinsically related to the renormalization process. As in dimension one, it is a Cantor set on which the map acts like the dyadic adding machine. The construction is similar to the construction of the limit set of an iterated function system. As a matter of fact, the Cantor set of the renormalization fixed point F_*, is the limit set of an iterated function system, see Figure 3.1.

The stable manifold $W^s(F_*) \subset \mathrm{Cons}_0(D)$, also called *the universality class*, of F_*, consists of the maps with $R^n F \to F_*$. Trough out this section we will consider a fixed map $F \in W^s(F_*)$.

We will use the notation

$$\psi_0^n = \Lambda_{R^{n-1}F} : D \to \mathbb{R}^2$$

and

$$\psi_1^n = R^{n-1}F \circ \Lambda_{R^{n-1}F} : D \to \mathbb{R}^2$$

Observe,

$$R^n F = (\psi_0^n)^{-1} \circ R^{n-1}F \circ \psi_0^n.$$

The convergence of $R^n F \to F_*$ and Theorem 2.1(8) immediately implies

Lemma 3.1. *For every* $F \in W^s(F_*)$

$$\lim_{n \to \infty} ||\psi_{0,1}^n(F) - \psi_{0,1}(F_*)||_{C^2} = 0.$$

Remark 3.1. Lemma 3.1 shows a crucial difference between conservative and dissipative infinitely renormalizable maps. In the conservative case, the rescaling maps converge to (non-degenerated) diffeomorphisms. In the dissipative case, the corresponding rescaling ψ_1^n converge to a degenerated map, a map with one-dimensional image. This degeneration is at the heart of the difficulties of the non-rigidity phenomena observed in dissipative maps, see [CLM]. The degeneration occurs in a universal manner. This universal degeneration is responsible for the probabilistic nature of geometry in dimension two where one observes Probabilistic Universality and Probabilistic Rigidity, see [LM3].

Remark 3.2. The convergence in Lemma 3.1 holds in any C^k-distance, $k \geq 0$. We only need control of the lower order derivatives.

Remark 3.3. The construction of the Cantor set in the conservative case is exactly the same as in the dissipative case. The difference lies in the asymptotic behavior of the rescalings, see remark 3.1.

Let

$$\Psi_{00}^2 = \psi_0^1 \circ \psi_0^2, \quad \Psi_{01}^2 = \psi_0^1 \circ \psi_1^2, \quad \Psi_{10}^2 = \psi_1^1 \circ \psi_0^2, \quad \dots.$$

and, proceeding this way, construct, for any $w = (w_1, \dots, w_n) \in \{0,1\}^n$, $n \geq 1$, the maps

$$\Psi_w^n = \psi_{w_1}^1 \circ \cdots \circ \psi_{w_n}^n : \mathrm{Dom}(R^n F) \to \mathrm{Dom}(F).$$

The notation $\Psi_w^n(F)$ will also be used to emphasize the map under consideration. The following Lemma follows directly from Proposition 2.3.

Lemma 3.2. *For every $F \in W^s(F_*)$ there exists $C > 0$ such that for any word $w \in \{0,1\}^n$, $n \geq 1$,*

$$\|D\Psi_w^n\| \leq C\theta_2^n$$

where $\theta_2 < 1$ is given in Proposition 2.3 and (2.5).

Define the *pieces* of the nth-*level* or n^{th}-*scale* as follows. They are closed topological disks. For each $w \in \{0,1\}^n$ let

$$B_{w0}^n \equiv B_{w0}^n(F) = \Psi_w^n(B_0(R^n F))$$

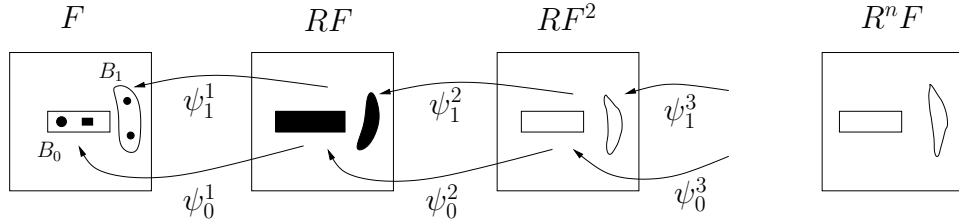and

$$B_{w1}^n \equiv B_{w1}^n(F) = \Psi_w^n(B_1(R^n F)).$$

FIGURE 3.1. The Renormalization Microscope

The set of words $\{0,1\}^n$ can be viewed as the additive group of residues $\bmod\, 2^n$ by letting

$$w \mapsto \sum_{k=0}^{n-1} w_{k+1} 2^k.$$

Let $p \colon \{0,1\}^n \to \{0,1\}^n$ be the operation of adding 1 in this group.

Lemma 3.3. *For every $F \in W^s(F_*)$ and $n \geq 1$*

1) *The above families of pieces are nested:*

$$B_{w\nu}^n \subset B_w^{n-1}, \quad w \in \{0,1\}^n, \ \nu \in \{0,1\}.$$

2) *The pieces B_w^n, $w \in \{0,1\}^{n+1} \setminus \{1^{n+1}\}$, are pairwise disjoint.*
3) *Under F, the pieces are permuted as follows.*

$$F(B_w^n) = B_{p(w)}^n,$$

unless $p(w) = 0^{n+1}$. If $p(w) = 0^{n+1}$, then $F(B_w^n) \cap B_{0^{n+1}}^n \neq \emptyset$.

Proof. The first assertion holds by construction: Let $w \in \{0,1\}^{n-1}$, $w_n, v \in \{0,1\}$, and use Lemma 2.2,

$$\begin{aligned} B_{ww_nv}^n = \Psi_{ww_n}^n(B_v(R^nF)) &= \Psi_w^{n-1} \circ \psi_{w_n}^n(B_v(R^nF)) \\ &\subset \Psi_w^{n-1}(B_{w_n}(R^{n-1}F)) = B_{ww_n}^{n-1}(F). \end{aligned}$$

The second follows by induction. For all maps under consideration we have the disjointness property (2.1) which says, that $B_0^1(F)$ and $B_1^1(F)$ are disjoint. Assume that the pieces of the nth generation are disjoint for all maps under consideration. This implies that the pieces $B_{0w}^{n+1} \subset B_0^1$, $w \in \{0,1\}^{n+1}$, are pairwise disjoint, as they are images of the disjoint pieces $B_w^n(RF)$ by the map ψ_1^0. Applying F, we see that the pieces $B_{1w}^{n+1} \subset B_1^1$, $w \in W^n$, are pairwise disjoint as well. The assertion follows because B_1^1 and B_0^1 are also disjoint.

Let us inductively check the third assertion. For $n = 1$, we have:

$$B_1^1 = F(B_0^1) \text{ and } F(B_1^1) \cap B_0^1 \neq \emptyset.$$

Consider now the pieces $B_w^n(RF)$, $w \in \{0,1\}^{n+1}$, of level n for RF. Assume inductively that they are permuted by RF as required. Then the pieces $B_{0w}^{n+1} = \psi_0^1(B_w^n(RF))$ are permuted in the same fashion under F^2. Moreover, $B_{1w}^{n+1} = \psi_1^1(B_w^n(RF)) = F(B_{0w}^{n+1})$, and the conclusion follows. $\square$

Furthermore, Lemma 3.2 implies:

Lemma 3.4. *For every $F \in W^s(F_*)$ there exists $C > 0$ such that for all $w \in \{0,1\}^{n+1}$, $\operatorname{diam} B_w^n \leq C\theta_2^n$.*

Let

$$\mathcal{O} \equiv \mathcal{O}_F = \bigcap_{n=1}^{\infty} \bigcup_{w \in \{0,1\}^n} B_w^n.$$

Let us also consider the *diadic group* $\{0,1\}^\infty = \varprojlim\{0,1\}^n$. The elements of $\{0,1\}^\infty$ are infinite sequences $(w_1w_2\ldots)$ of symbols 0 and 1

that can be also represented as formal power series

$$w \mapsto \sum_{k=0}^{\infty} w_{k+1} 2^k.$$

The integers $\mathbb{Z}$ are embedded into $\{0,1\}^{\infty}$ as finite series. The *adding machine* $p : \{0,1\}^{\infty} \to \{0,1\}^{\infty}$ is the operation of adding 1 in this group. The discussion above yields:

Theorem 3.5. *For every $F \in W^s(F_*)$ the set $\mathcal{O}_F$ is an invariant Cantor set. The map F acts on $\mathcal{O}_F$ as the dyadic adding machine $p : \{0,1\}^{\infty} \to \{0,1\}^{\infty}$. The conjugacy between p and $F|\mathcal{O}_F$ is given by the following homeomorphism $h_F : \{0,1\}^{\infty} \to \mathcal{O}_F$:*

$$(3.1) \qquad h_F : w = (w_1 w_2 \dots) \mapsto \bigcap_{n=1}^{\infty} B^n_{w_1 \dots w_{n+1}}.$$

Furthermore, $\mathcal{O}_F$ has Lebesgue measure zero with

$$HD(\mathcal{O}_F) \leq -\frac{\log 2}{\log \theta_2} \leq 0.795.$$

The invariant Cantor sets $\mathcal{O}_F$ are the counterpart of the period doubling Cantor sets in one-dimensional dynamics and strongly dissipative Hénon-like maps, see [CLM, GST, Mi]. The dynamics of F restricted to $\mathcal{O}_F$ is conjugated to the adding machine. The adding machine is uniquely ergodic. Let μ be the unique invariant measure of F restricted to $\mathcal{O}_F$:

$$\mu(B^n_w) = \frac{1}{2^{n+1}}.$$

Theorem 3.6. *The measure μ_F of every map $F \in W^s(F_*)$ has a single characteristic exponent, $\chi = 0$.*

Proof. The largest characteristic exponent is denoted by χ. Let F_n be the n-th renormalization of F. This map restricted to $B_0(R^n F) \cup B_1(R^n F)$ is smoothly conjugate to the restriction of F^{2^n} to the piece $B^n_{0^n 0} \cup B^n_{0^n 1}$. Let μ_n be the normalized restriction of μ to $B^n_{0^n 0} \cup B^n_{0^n 1}$, and let ν_n be the invariant measure on the Cantor set of F_n. Note that these two measures are preserved by the conjugacy. Then

$$2^n \chi = \chi(F^{2^n} | B^n_{0^n 0} \cup B^n_{0^n 1}, \mu_n) = \chi(F_n, \nu_n) \leq \int \log \|DF_n\| \, d\nu_n \leq C,$$

since the maps $F_n \to F_*$ have uniformly bounded C^1-norms.

Hence, $\chi \leq 0$. If $\chi < 0$, both characteristic exponents of F would be negative. This contradicts the relation $\chi + \chi_- = 0$ which holds because the map preserves the Lebesgue measure. $\qquad \square$

4. Rigidity

Let $\mathcal{O} \subset \mathbb{R}^2$. A map $h : \mathcal{O} \to \mathbb{R}^2$ is *differentiable* at $x_0 \in \mathcal{O}$ if there exists a linear map $Dh(x_0) : \mathbb{R}^2 \to \mathbb{R}^2$ such that for $x \in \mathcal{O}$

$$h(x) = h(x_0) + Dh(x_0)(x - x_0) + o(|x - x_0|).$$

If the map $x \mapsto Dh(x)$ is C^α then we say that h is a $C^{1+\alpha}$-map and if it is Lipschitz ($\alpha = 1$) we say that h is $C^{1+\mathrm{Lip}}$. Observe, the composition of two $C^{1+\alpha}$-maps is again $C^{1+\alpha}$.

A bijection $h : \mathcal{O} \to h(\mathcal{O}) \subset \mathbb{R}^2$ is a $C^{1+\alpha}$-*diffeomorphism* if there exists a map $h^{-1} : h(\mathcal{O}) \to \mathbb{R}^2$ with $h^{-1} \circ h = \mathrm{id}$ and both maps are $C^{1+\alpha}$.

There are many conjugations between $\mathcal{O}_F$ and $\mathcal{O}_{\tilde{F}}$ with $F, \tilde{F} \in W^s(F_*)$. However, we will only consider conjugations $h : \mathcal{O}_F \to \mathcal{O}_{\tilde{F}}$ defined by

$$h = h_{\tilde{F}} \circ h_F^{-1},$$

where h_F and $h_{\tilde{F}}$ are the conjugations from $\{0,1\}^\infty$ to $\mathcal{O}_F$ and $\mathcal{O}_{\tilde{F}}$ respectively, see Theorem 3.5.

Definition 4.1. The Cantor set $\mathcal{O}_F$ is *rigid* if for some $\alpha > 0$, the conjugation $h : \mathcal{O}_F \to \mathcal{O}_{F_*}$ is a $C^{1+\alpha}$-diffeomorphsim. Notation, $\mathcal{O}_F = \mathcal{O}_{F_*} \mathrm{mod}(C^{1+\alpha})$.

Let

$$(4.1) \qquad \alpha_0 = \frac{\ln \theta_2 \nu}{\ln \theta_1} - 1 > 0.237.$$

Theorem 4.1. *The Cantor set $\mathcal{O}_F$, with $F \in W_{loc}^s(F_*)$, is rigid. Namely,*

$$\mathcal{O}_F = \mathcal{O}_{F_*} \mathrm{mod}(C^{1+\alpha})$$

for every $0 < \alpha < \alpha_0$.

The proof of the Rigidity Theorem 4.1 consists of two parts. The main part, Proposition 4.5 treats the rigidity question for maps in the local strong stable manifold. The other reduces the rigidity question to this Proposition, using that the local weak stable manifold is part of a smooth conjugacy class, see Lemma 4.3.

Lemma 4.2. *If $\mathcal{O}_{R^n F} = \mathcal{O}_{F_*} \mathrm{mod}(C^{1+\alpha})$ then $\mathcal{O}_F = \mathcal{O}_{F_*} \mathrm{mod}(C^{1+\alpha})$.*

Proof. Assume that the conjugation $\hat{h}_n : \mathcal{O}_{R^n F} \to \mathcal{O}_{F_*}$ is a $C^{1+\alpha}$-diffeomorphism. Use the notation $\mathcal{O}_w^k = B_w^k \cap \mathcal{O}_F$ (To indicate the map under consideration we will also use $\mathcal{O}_w^k(F) = B_w^k \cap \mathcal{O}_F$) . Observe, for every word $w \in \{0,1\}^{k+1}$ one has

$$(4.2) \qquad \hat{h}_n(\mathcal{O}_w^k(R^n F)) = \mathcal{O}_w^k(F_*).$$

Define the following $C^{1+\alpha}$-diffeomorphism $h_n : \mathcal{O}(F) \to \mathcal{O}(\tilde{F})$. For $w \in \{0,1\}^n$, $v \in \{0,1\}$ let

$$(4.3) \qquad h_n|\mathcal{O}^n_{wv} = \Psi^n_w(F_*) \circ \hat{h}_n \circ (\Psi^n_w(F))^{-1} : \mathcal{O}^n_{wv}(F) \to \mathcal{O}^n_{wv}(\tilde{F}).$$

From (4.2) we get a similar property for h_n: for every word $w \in \{0,1\}^{k+1}$, $k \geq n$,

$$h_n(\mathcal{O}^k_w(F)) = \mathcal{O}^k_w(F_*).$$

Let $x \in \mathcal{O}_F$, say $x = \bigcap_k \mathcal{O}^k_{w^k}(F)$, with $w^k \in \{0,1\}^k$. Observe,

$$F(x) = \bigcap_k \mathcal{O}^k_{p(w^k)}(F)$$

and

$$h(x) = \bigcap_k \mathcal{O}^k_{w^k}(F_*).$$

Hence,

$$F_*(h(x)) = \bigcap_k \mathcal{O}^k_{p(w^k)}(F_*)$$

and

$$h(F(x)) = \bigcap_k \mathcal{O}^k_{p(w^k)}(F_*).$$

Indeed, the diffeomorphism is a conjugation, $h \circ F = F_* \circ h$. $\qquad\square$

Consider the family of diffeomorpsims $h_t : D \to h_t(D) \subset \mathbb{R}^2$ given in Theorem 2.1(7).

Lemma 4.3. *There exists $c > 0$ such that for any $F \in W^s_{loc}(F_*)$ close enough to F_* the family $F_t = h_t^{-1} \circ F \circ h_t$, $|t| \leq c$, has a unique intersection with the strong stable manifold $W^{ss}_{loc}(F_*)$.*

Proof. This Lemma follows immediately from the transversality property mentioned in Theorem 2.1(7). $\qquad\square$

Lemma 4.4. *For every $F \in W^s(F_*)$ there exists $\tilde{F} \in W^{ss}_{loc}(F_*)$ such that for some $n \geq 1$, the conjugation $h : \mathcal{O}_{R^n F} \to \mathcal{O}_{\tilde{F}}$ is a C^{1+Lip}-diffeomorphism.*

Proof. For $n \geq 1$ large enough, $R^n F \in W^s_{loc}(F_*)$ is close enough to F_* to apply Lemma 4.3. $\qquad\square$

Remark 4.1. Observe, the conjugation in lemma 4.4 is much better than $C^{1+\mathrm{Lip}}$. It is in fact the restriction of a rational map.

Proposition 4.5. *The Cantor set $\mathcal{O}_F$, with $F \in W^{ss}_{loc}(F_*)$, is rigid. Namely,*

$$\mathcal{O}_F = \mathcal{O}_{F_*} \mathrm{mod}(C^{1+\alpha})$$

for every $0 < \alpha < \alpha_0$.

The proof of this key Proposition will be presented in a series of Lemmas. Fix two maps $F, \tilde{F} \in W^{ss}_{\text{loc}}(F_*)$. The constants which will appear in the following analysis are independent of these maps. They only depend on the size of $W^{ss}_{\text{loc}}(F_*)$.

Consider the conjugation $h : \mathcal{O}_F \to \mathcal{O}_{\tilde{F}}$ defined by

$$h = h_{\tilde{F}} \circ h_F^{-1},$$

where h_F and $h_{\tilde{F}}$ are the conjugations from the adding machine $\{0,1\}^\infty$ to $\mathcal{O}_F$ and $\mathcal{O}_{\tilde{F}}$ respectively, see Theorem 3.5. We will show that this map is $C^{1+\alpha}$, for any $\alpha > 0$ satisfying

(4.4)
$$\frac{\theta_2 \nu}{\theta_1^{1+\alpha}} < 1$$

or equivalently $0 < \alpha < \alpha_0$, see (4.1).

Proposition 2.3 has an unfortunate form. Ideally, we would have

$$\theta_1 \cdot |v| \le |D\psi_{0,1}(x, u)v| \le \theta_2 \cdot |v|.$$

However, this only holds on a small neighborhood of the Cantor sets. This inconvenience forces us to work in "steps of four".

Let $\mathcal{O}_n = \bigcup_{w \in \{0,1\}^{4n+1}} B_w^{4n}(F)$ and similarly define $\tilde{\mathcal{O}}_n$ for $\tilde{F}$ (In the sequel we will use a tilde to indicate whether an object refers to F or $\tilde{F}$). The map $h_n : \mathcal{O}_n \to \tilde{\mathcal{O}}_n$ is defined by

$$h_n|B_{wv}^{4n} = \tilde{\Psi}_w^{4n} \circ (\Psi_w^{4n})^{-1},$$

where $w \in \{0,1\}^{4n}$ and $v \in \{0,1\}$, $\Psi_w^{4n} = \Psi_w^{4n}(F)$ and $\tilde{\Psi}_w^{4n} = \Psi_w^{4n}(\tilde{F})$. This map h_n is an approximate conjugation, see Figure 4.1. Compare this map with the conjugation (4.3). The conjugation $\hat{h}_n$ between the nth-renormalization is replaced here by the identity. To indicate the maps under consideration we will also use the notation $h_n(F, \tilde{F})$.

Observe, $h_n|\mathcal{O}_F \to h$, where $h : \mathcal{O}_F \to \mathcal{O}_{\tilde{F}}$ is the conjugation. The following Lemmas will describe this convergence in more detail. The proofs are by induction. The idea of this induction is the following. The maps h_n are quasi conjugations up to the $4n^{th}$-scale. We use the uniform control of these maps at the $4n^{th}$-scale for the renormalizations and push it down to the next scale of the original systems. The control of the derivative of the quasi-conjugations, see Lemma 4.7, Lemma 4.8, and Lemma 4.9 relies in an essential way on the fact that the renormalizations $R^4 F$ and $R^4 \tilde{F}$ of F and $\tilde{F}$ are strictly closer, see Theorem 2.1(6), than the original systems. Namely, this implies that their
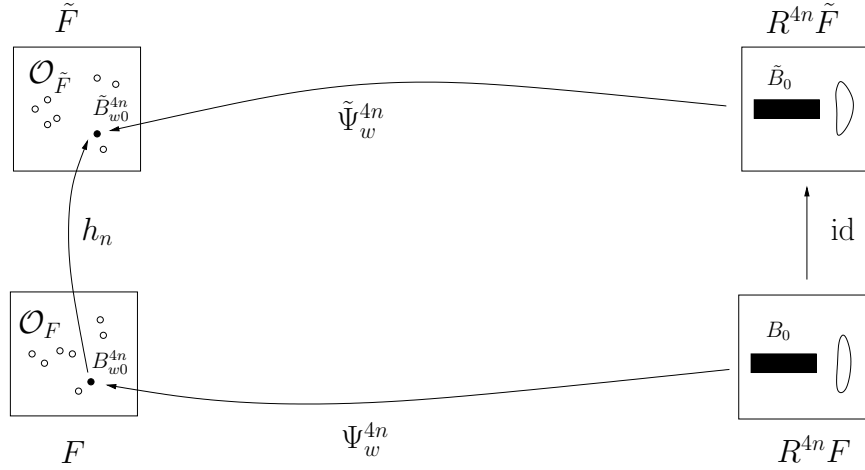
FIGURE 4.1. The Aproximate Conjugation h_n

quasi-conjugation at the $4n^{th}$-scale is strictly better than the quasi-conjugation at the nth-scale of the original systems. This effect causes the contraction in for example the estimate (4.20).

Lemma 4.6. *There exists a constant $C > 0$ such that*

$$|h_{n+1}(x) - h_n(x)| \leq C \cdot (\theta_2 \nu)^{4n} \cdot d(F, \tilde{F}),$$

with $x \in \mathcal{O}_{n+1}$. In particular,

$$|h_n(x) - x| \leq C \cdot d(F, \tilde{F})$$

Proof. The proof will be by induction. Observe, $h_0(x) = x$ and

$$h_1(x) = \tilde{\Psi}_w^4 \circ (\Psi_w^4)^{-1}(x),$$

when $x \in B_w^4$ with $w \in \{0,1\}^{4+1}$. Because the maps $\Psi_w^4(F)$ depend analytically on F, see Theorem 2.1(8), we get

$$|h_1(x) - h_0(x)| \leq C \cdot d(F, \tilde{F}).$$

Assume the Lemma holds for n. Let $\hat{h}_n = h_n(R^4 F, R^4 \tilde{F})$. Given a point in the domain of h_{n+2}, say $x \in B_{wv}^{4(n+2)}$ with $w \in \{0,1\}^4$ and $v \in \{0,1\}^{4(n+1)+1}$. Then

$$h_{k+1}(x) = \tilde{\Psi}_w^4 \circ \hat{h}_k \circ (\Psi_w^4)^{-1}(x),$$

with either $k = n$ or $k = n + 1$. Also observe that $(\Psi_w^4)^{-1}(B_{wv}^{4(n+2)}) \subset B_v^{4(n+1)} \subset B_0(R^4 F) \cup B_1(R^4 F)$. This will allow us to apply Proposition 2.3. Let $x_- = (\Psi_w^4)^{-1}(x)$. Use Proposition 2.3, Theorem 2.1(6), and

the induction hypothesis, to obtain

$$|h_{n+2}(x) - h_{n+1}(x)| \leq |\tilde{\Psi}_w^4 \circ \hat{h}_{n+1}(x_-) - \tilde{\Psi}_w^4 \circ \hat{h}_n(x_-)|$$
$$\leq \theta_2^4 \cdot |\hat{h}_{n+1}(x_-) - \hat{h}_n(x_-)|$$
$$\leq \theta_2^4 \cdot C \cdot (\theta_2 \nu)^{4n} \cdot d(R^4 F, R^4 \tilde{F})$$
$$\leq C \cdot (\theta_2 \nu)^{4(n+1)} \cdot d(F, \tilde{F}).$$

The proximity to identity follows from $h_0 = \mathrm{id}$. $\qquad\square$

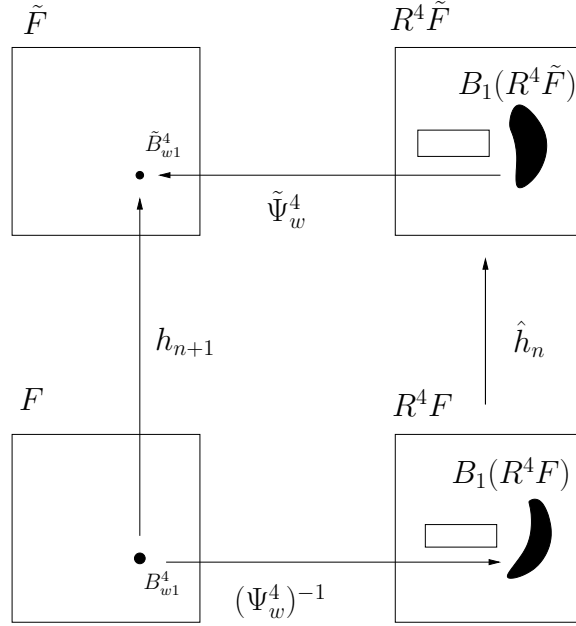Observe that the maps h_n are differentiable, in fact analytic.

FIGURE 4.2. The Induction Step for h_{n+1}

Lemma 4.7. *There exists a constant $C > 0$ such that*

$$|Dh_{n+1}(x) - Dh_n(x)| \leq C \cdot \left(\frac{\theta_2 \nu}{\theta_1}\right)^{4n} \cdot d(F, \tilde{F}),$$

with $x \in \mathcal{O}_{n+1}$. In particular,

$$|Dh_n(x) - \mathrm{id}\,| \leq C \cdot d(F, \tilde{F}).$$

Proof. The proof will be by induction. Observe, $Dh_0(x) = \mathrm{id}$ and

$$h_1(x) = \tilde{\Psi}_w^4 \circ (\Psi_w^4)^{-1}(x),$$

when $x \in B_{wv}^4$ with $w \in \{0,1\}^4$ and $v \in \{0,1\}$. Because the maps $\Psi_w^4(F)$ depend analytically on F, see Theorem 2.1(8), we get

$$|Dh_1(x) - Dh_0(x)| \leq C \cdot d(F, \tilde{F}).$$

Assume the Lemma holds for $0, 1, 2, \ldots, n$ with constant $C_n > 0$. This implies

$$(4.5) \qquad |Dh_n(x) - \mathrm{id}\,| \leq C \cdot C_n \cdot d(F, \tilde{F}),$$

where we used $Dh_0 = \mathrm{id}$.

Let $\hat{h}_n = h_n(R^4 F, R^4 \tilde{F})$. Given a point in the domain of h_{n+2}, say $x \in B_{wv}^{4(n+2)}$ with $w \in \{0,1\}^4$ and $v \in \{0,1\}^{4(n+1)+1}$. Then

$$Dh_{k+1}(x) = D\tilde{\Psi}_w^4(\hat{x}_k) \circ D\hat{h}_k(x_-) \circ D(\Psi_w^4)^{-1}(x),$$

where $x_- = (\Psi_w^4)^{-1}(x)$ and $\hat{x}_k = \hat{h}_k(x_-)$ and either $k = n$ or $k = n+1$. Also observe that $(\Psi_w^4)^{-1}(B_{wv}^{4(n+2)}) \subset B_v^{4(n+1)} \subset B_0(R^4 F) \cup B_1(R^4 F)$. This will allow us to apply Proposition 2.3. Let $d = d(F, \tilde{F})$. Use Proposition 2.3, Theorem 2.1(6), the induction hypothesis, Lemma 4.6, and (4.5) to obtain

$$|Dh_{n+2}(x) - Dh_{n+1}(x)|$$

$$\leq |D\tilde{\Psi}_w^4(\hat{x}_{n+1})D\hat{h}_{n+1}(x_-)D(\Psi_w^4)^{-1}(x) - D\tilde{\Psi}_w^4(\hat{x}_n)D\hat{h}_n(x_-)D(\Psi_w^4)^{-1}(x)|$$

$$\leq |D\tilde{\Psi}_w^4(\hat{x}_n)| \cdot |D\hat{h}_{n+1}(x_-) - D\hat{h}_n(x_-)| \cdot |D(\Psi_w^4)^{-1}(x)| +$$

$$\quad |D\tilde{\Psi}_w^4(\hat{x}_{n+1}) - D\tilde{\Psi}_w^4(\hat{x}_n)| \cdot |D\hat{h}_{n+1}(x_-)| \cdot |D(\Psi_w^4)^{-1}(x)|$$

$$\leq \left(\frac{\theta_2}{\theta_1}\right)^4 \cdot C_n \cdot \left(\frac{\theta_2 \nu}{\theta_1}\right)^{4n} \cdot d(R^4 F, R^4 \tilde{F}) + C \cdot |\hat{x}_{n+1} - \hat{x}_n| \cdot |D\hat{h}_{n+1}(x_-)|$$

$$\leq C_n \cdot \left(\frac{\theta_2 \nu}{\theta_1}\right)^{4(n+1)} \cdot d + C \cdot (\theta_2 \nu)^{4n} \cdot (1 + C \cdot C_n \cdot d) \cdot d$$

This implies

$$C_{n+1} \leq C_n + C \cdot \theta_1^{4n} \cdot (1 + C \cdot C_n).$$

In particular, the sequence C_n is bounded. The Lemma is proved. $\square$

Lemma 4.8. *For every $0 < \alpha < \alpha_0$ there exists a constant $C > 0$ such that*

$$|h_n(y) - h_n(x) - Dh_n(x)(y - x)| \leq C \cdot d(F, \tilde{F}) \cdot |y - x|^{1+\alpha},$$

where $x, y \in \mathcal{O}_n$.

Proof. Fix $< \alpha < \alpha_0$. Consider the collection of diffeomorphisms

$$\{\Psi_w^4(F) | F \in W_{\mathrm{loc}}^{ss}(F_*) \text{ and } w \in \{0,1\}^4\}$$

and
$$\{(\Psi_w^4(F))^{-1}|F \in W_{\text{loc}}^{ss}(F_*) \text{ and } w \in \{0,1\}^4\}.$$
Let $K > 0$ be the constant such that the Lemmas 5.1, 5.2, 5.3, and 5.4 hold for these collections. Furthermore, choose $\delta_0 > 0$ such that

$$(4.6) \qquad \gamma = \left(\frac{\theta_2\nu}{\theta_1^{1+\alpha}}\right)^4 \cdot (1 + K\delta_0)^{1+\alpha} < 1.$$

We may assume that for every $F \in W_{\text{loc}}^{ss}(F_*)$ and $w_1, w_2 \in \{0,1\}^{4+1}$
$$\text{dist}(B_{w_1}^4(F), B_{w_2}^4(F)) \geq \delta_0.$$

Use the notation $d = d(F, \tilde{F})$.

The proof is by induction. The Lemma holds for $n = 0$ because $h_0 = \text{id}$. It holds even with $C_0 = 0$. But we will choose $C_0 > 0$ large enough such that the Lemma holds with $C = C_0$ whenever $|y - x| \geq \delta_0$. This is indeed possible because from Lemma 4.7 we get a uniform bound on the norm of Dh_n.

Assume the Lemma holds for n:
$$|h_n(y) - h_n(x) - Dh_n(x)(y - x)| \leq C_n \cdot d \cdot |y - x|^{1+\alpha}.$$

Choose $x, y \in \mathcal{O}_{n+1}$ with $|y - x| \leq \delta_0$. Then there is $w \in \{0,1\}^4$, and $v \in \{0,1\}$, with $x, y \in B_{wv}^4$. Let
$$\psi = \Psi_w^4(F)$$
and
$$\tilde{\psi} = \Psi_w^4(\tilde{F}).$$
Observe, $\psi^{-1}(x), \psi^{-1}(y) \in B_0(R^4 F) \cup B_1(R^4 F)$. This will allow us to apply Proposition 2.3. The maps $\Psi_w^4(F)$ depend analytically on F, see Theorem 2.1(8). Hence,

$$(4.7) \qquad\qquad ||\psi - \tilde{\psi}||_{C^2} \leq C \cdot d.$$

Let
$$\hat{h}_n = h_n(R^4 F, R^4 \tilde{F})$$
then
$$h_{n+1}(x) = \tilde{\psi} \circ \hat{h}_n \circ \psi^{-1}(x)$$
and
$$h_{n+1}(y) = \tilde{\psi} \circ \hat{h}_n \circ \psi^{-1}(y).$$
Our goal is to estimate $|h_{n+1}(y) - h_{n+1}(x) - Dh_{n+1}(x)(y - x)|$, the distortion (The appendix gives some estimates on distortions in general). To avoid very cumbersome expressions we will work backward.

The maps ψ and $\tilde{\psi}$ will contribute to the distortion. The first part of the proof will lead to an estimate on the influence of these distortions, see (4.14). From Lemma 4.6 we get

$$(4.8) \qquad \hat{h}_n(\psi^{-1}(x)) = \psi^{-1}(x) + \Delta$$

with

$$(4.9) \qquad |\Delta| \leq C \cdot d$$

where we used $d(R^4 F, R^4 \tilde{F}) \leq \nu^4 d$, see Theorem 2.1(6). From Lemma 4.7 we get

$$(4.10) \qquad D\hat{h}_n(\psi^{-1}(x)) = \mathrm{id} + O(d).$$

Use (4.8) and (4.10) and Lemma 5.3 in the following estimates. Let

$$\begin{aligned}
z &= \hat{h}_n(\psi^{-1}(x)) + D\hat{h}_n(\psi^{-1}(x))(D\psi^{-1}(x)(y - x) + E_{\psi^{-1}}(x,y)) \\
&= \psi^{-1}(x) + D\psi^{-1}(x)(y - x) + E_{\psi^{-1}}(x,y) + \Delta + O(d|y - x|).
\end{aligned}$$

Hence,

$$(4.11) \qquad z = \psi^{-1}(y) + \Delta + O(d|y - x|).$$

The next term we treat, using (4.8), (4.11), Lemma 5.4, (4.7), Lemma 5.2, is

$$\begin{aligned}
E_{\tilde{\psi}}(\hat{h}_n(\psi^{-1}(x)), z) &= E_{\tilde{\psi}}(\psi^{-1}(x) + \Delta, \psi^{-1}(y) + \Delta + O(d|y - x|)) \\
&= E_\psi(\psi^{-1}(x) + \Delta, \psi^{-1}(y) + \Delta) + O(d|y - x|^2)
\end{aligned}$$

Hence, using Lemma 5.1, and (4.9) we obtain

$$(4.12) \qquad E_{\tilde{\psi}}(\hat{h}_n(\psi^{-1}(x)), z) = E_\psi(\psi^{-1}(x), \psi^{-1}(y)) + O(d|y - x|^2).$$

The next term we treat, using (4.7), Lemma 4.7, (4.8), and (4.9), is

$$\begin{aligned}
&D\tilde{\psi}(\hat{h}_n(\psi^{-1}(x))) \cdot D\hat{h}_n(\psi^{-1}(x)) \cdot E_{\psi^{-1}}(x,y) = \\
&(D\psi(\hat{h}_n(\psi^{-1}(x))) + O(d)) \cdot (\mathrm{id} + O(d)) \cdot E_{\psi^{-1}}(x,y) = \\
&(D\psi(\psi^{-1}(x)) + O(d)) \cdot (\mathrm{id} + O(d)) \cdot E_{\psi^{-1}}(x,y).
\end{aligned}$$

Hence, using Lemma 5.3,

$$(4.13) \qquad \begin{aligned}
&D\tilde{\psi}(\hat{h}_n(\psi^{-1}(x)))D\hat{h}_n(\psi^{-1}(x))E_{\psi^{-1}}(x,y) = \\
&D\psi((\psi^{-1}(x))E_{\psi^{-1}}(x,y) + O(d|y - x|^2).
\end{aligned}$$

Now we apply the cancellation from Lemma 5.5 to (4.12) and (4.13) to obtain an estimate on the influence of the distortions of ψ and $\tilde{\psi}$

$$(4.14) \quad D\tilde{\psi}(\hat{h}_n(x_-))D\hat{h}_n(x_-)E_{\psi^{-1}}(x,y) + E_{\tilde{\psi}}(\hat{h}_n(x_-), z) = O(d|y - x|^2),$$

where $x_- = \psi^{-1}(x)$.

The second part of the proof will give an estimate on the influence of the distortion of the approximate conjugation $\hat{h}_n$ between $R^4 F$ and $R^4 \tilde{F}$. First use the induction hypothesis, to obtain

$$
(4.15) \quad
\begin{aligned}
\hat{h}_n(\psi^{-1}(y)) &= \hat{h}_n(\psi^{-1}(x) + D\psi^{-1}(x)(y - x) + E_{\psi^{-1}}(x, y)) \\
&= z + \Delta z,
\end{aligned}
$$

where

$$(4.16) \quad |\Delta z| \leq C_n \cdot d(R^4 F, R^4 \tilde{F}) \cdot |D\psi^{-1}(x)(y - x) + E_{\psi^{-1}}(x, y)|^{1+\alpha}.$$

Now use Theorem 2.1(6), Proposition 2.3, Lemma 5.3, to obtain

$$(4.17) \quad |\Delta z| \leq C_n \cdot \left(\frac{\nu}{\theta_1^{1+\alpha}} \right)^4 \cdot d \cdot |y - x|^{1+\alpha} \cdot (1 + K|y - x|)^{1+\alpha}.$$

Hence, again using Proposition 2.3, (4.6), and $|y - x| \leq \delta_0$,

$$(4.18) \quad |\tilde{\psi}(z + \Delta z) - \tilde{\psi}(z)| \leq C_n \cdot \gamma \cdot d \cdot |y - x|^{1+\alpha}.$$

The final term is

$$
\begin{aligned}
\tilde{\psi}(z) =\,& h_{n+1}(x)+ \\
& D\tilde{\psi}(\hat{h}_n(x_-)) \cdot D\hat{h}_n(x_-) \cdot \{D\psi^{-1}(x)(y - x) + E_{\psi^{-1}}(x, y)\}+ \\
& E_{\tilde{\psi}}(\hat{h}_n(\psi^{-1}(x)), z) \\
=\,& h_{n+1}(x) + Dh_{n+1}(x)(y - x)+ \\
& D\tilde{\psi}(\hat{h}_n(x_-)) \cdot D\hat{h}_n(x_-) \cdot E_{\psi^{-1}}(x, y) + E_{\tilde{\psi}}(\hat{h}_n(x_-), z).
\end{aligned}
$$

From (4.14) we obtain

$$(4.19) \quad \tilde{\psi}(z) = h_{n+1}(x) + Dh_{n+1}(x)(y - x) + O(d|y - x|^2).$$

Finally, we will use both parts of the proof to estimate the distortion of h_{n+1}. Using (4.15) we can estimate

$$h_{n+1}(y) = \tilde{\psi}(z + \Delta z) = \tilde{\psi}(z) + (\tilde{\psi}(z + \Delta z) - \tilde{\psi}(z))$$

by (4.18) and (4.19), we obtain

$$
\begin{aligned}
|h_{n+1}(y) - h_{n+1}(x) - Dh_{n+1}(x)(y - x)| &\leq \\
C_n \cdot \gamma \cdot d \cdot |y - x|^{1+\alpha} + C \cdot d \cdot |y - x|^2 &= \\
(C_n \gamma + C) \cdot d \cdot |y - x|^{1+\alpha}.&
\end{aligned}
$$

Hence,

$$(4.20) \quad C_{n+1} \leq \max\{C_0, \gamma \cdot C_n + C\}.$$

This finishes the proof of the Lemma. $\qquad \square$

Remark 4.2. The application of Lemma 4.8 does not need the factor $d(F, \tilde{F})$. However, this factor plays a crucial role. The contraction in (4.20) by the factor $\gamma < 1$ arises in the equation (4.16), (4.17), and (4.18). The distance factor reveals the contraction.

　The proof became lengthy. The cause is the error term in (4.19). This is the sum of the distortions of $\tilde{\psi}$ and ψ^{-1}. The estimates show that these distortions essentially cancel each other. Without evoking the cancelation in (4.14) the error term would be of the order $|y - x|^2$ without the necessary factor $d(F, \tilde{F})$. Similar analysis occurs in the proof of Lemma 4.9.

Lemma 4.9. *For every $0 < \alpha < \alpha_0$ there exists a constant $C > 0$ such that*
$$|Dh_n(y) - Dh_n(x)| \le C \cdot d(F, \tilde{F}) \cdot |y - x|^\alpha,$$
where $x, y \in \mathcal{O}_n$.

Proof. Fix $< \alpha < \alpha_0$. Consider the collection of diffeomorphisms
$$\{\Psi_w^4(F) | F \in W_{\text{loc}}^{ss}(F_*) \text{ and } w \in \{0, 1\}^4\}$$
and
$$\{(\Psi_w^4(F))^{-1} | F \in W_{\text{loc}}^{ss}(F_*) \text{ and } w \in \{0, 1\}^4\}.$$
The Lemmas 5.6, 5.7, 5.8, and 5.9 hold uniformly for these collections. Choose $\delta_0 > 0$ such that for every $F \in W_{\text{loc}}^{ss}(F_*)$ and $w_1, w_2 \in \{0, 1\}^{4+1}$
$$\text{dist}(B_{w_1}^4(F), B_{w_2}^4(F)) \ge \delta_0.$$
Use the notation $d = d(F, \tilde{F})$.

　The proof is by induction. The Lemma holds for $n = 0$ because $h_0 = \text{id}$. It holds even with $C_0 = 0$. But we will choose $C_0 > 0$ large enough such that the Lemma holds with $C = C_0$ whenever $|y - x| \ge \delta_0$. This is indeed possible because from Lemma 4.7 we get a uniform bound on the norm of Dh_n.

　Assume the Lemma holds for n:
$$|Dh_n(y) - Dh_n(x)| \le C_n \cdot d \cdot |y - x|^\alpha.$$
Choose $x, y \in \mathcal{O}_{n+1}$ with $|y - x| \le \delta_0$. Then there is $w \in \{0, 1\}^4$, $v \in \{0, 1\}$, with $x, y \in B_{wv}^4$. Let
$$\psi = \Psi_w^4(F)$$
and
$$\tilde{\psi} = \Psi_w^4(\tilde{F}).$$

Observe, $\psi^{-1}(x), \psi^{-1}(y) \in B_0(R^4 F) \cup B_1(R^4 F)$. This will allow us to apply Proposition 2.3. Let

$$\hat{h}_n = h_n(R^4 F, R^4 \tilde{F})$$

then

$$Dh_{n+1}(x) = D\tilde{\psi}(\hat{h}_n(\psi^{-1}(x)))D\hat{h}_n(\psi^{-1}(x))D\psi^{-1}(x)$$

and

$$Dh_{n+1}(y) = D\tilde{\psi}(\hat{h}_n(\psi^{-1}(y)))D\hat{h}_n(\psi^{-1}(y))D\psi^{-1}(y).$$

Our goal is to estimate $|Dh_{n+1}(y) - Dh_{n+1}(x)|$. We will work backward again. The maps ψ and $\tilde{\psi}$ will contribute. The first part of the proof will lead to an estimate on this influence, see (4.24).

Use (4.8), (4.10), Lemma 4.8, and Proposition 2.3 in the following estimates. Let $x_- = \psi^{-1}(x)$ then

$$\hat{h}_n(\psi^{-1}(y)) = \hat{h}_n(x_-) + D\hat{h}_n(x_-)(\psi^{-1}(y) - \psi^{-1}(x)) + O(d \cdot |y - x|^{1+\alpha})$$
$$= \psi^{-1}(x) + \Delta + (\psi^{-1}(y) - \psi^{-1}(x)) + O(d|y - x|).$$

Hence,

$$(4.21) \qquad \hat{h}_n(\psi^{-1}(y)) = \psi^{-1}(y) + \Delta + O(d|y - x|).$$

Let $y_- = \psi^{-1}(y)$ then, by using (4.21), (4.8), Lemma 5.9, Lemma 5.6, and (4.9)

$$(4.22) \qquad ED_{\tilde{\psi}}(\hat{h}_n(x_-), \hat{h}_n(y_-)) = ED_\psi(x_-, y_-) + O(d|y - x|).$$

Use (4.8), (4.9), Lemma 5.8

$$(4.23) \;\; D\tilde{\psi}(\hat{h}_n(x_-))ED_{\psi^{-1}}(x, y) = D\psi(x_-)ED_{\psi^{-1}}(x, y) + O(d|y - x|).$$

The influence of ψ and $\tilde{\psi}$ is estimated by using (4.22) and (4.23)

$$D\tilde{\psi}(\hat{h}_n(x_-))ED_{\psi^{-1}}(x, y) + ED_{\tilde{\psi}}(\hat{h}_n(x_-), \hat{h}_n(y_-))D\psi^{-1}(y)$$
$$= D\psi(x_-)ED_{\psi^{-1}}(x, y) + ED_\psi(x_-, y_-)D\psi^{-1}(y) + O(d|y - x|).$$

Hence, by using the cancellation Lemma 5.10,

$$(4.24) \;\; D\tilde{\psi}(\tilde{x}_-)ED_{\psi^{-1}}(x, y) + ED_{\tilde{\psi}}(\tilde{x}_-, \tilde{y}_-))D\psi^{-1}(y) = O(d|y - x|),$$

where $\tilde{x}_- = \hat{h}_n(x_-)$ and $\tilde{y}_= \hat{h}_n(y_-)$.

Finally, using Lemma 4.7, lemma 5.8, and (4.24), we are prepared to estimate

$$\begin{aligned}
Dh_{n+1}(y) &= \\
&= D\tilde{\psi}(\tilde{y}_-)D\hat{h}_n(\psi^{-1}(y))D\psi^{-1}(y) \\
&= (D\tilde{\psi}(\tilde{x}_-) + ED_{\tilde{\psi}}(\tilde{x}_-, \tilde{y}_-))D\hat{h}_n(y_-)(D\psi^{-1}(x) + ED_{\psi^{-1}}(x, y)) \\
&= D\tilde{\psi}(\tilde{x}_-)D\hat{h}_n(y_-)D\psi^{-1}(x) + D\tilde{\psi}(\tilde{x}_-)(\mathrm{id} + O(d))ED_{\psi^{-1}}(x, y) + \\
&\quad ED_{\tilde{\psi}}(\tilde{x}_-, \tilde{y}_-)(\mathrm{id} + O(d))D\psi^{-1}(y) \\
&= D\tilde{\psi}(\tilde{x}_-)D\hat{h}_n(y_-)D\psi^{-1}(x) + O(d|y - x|).
\end{aligned}$$

Now, use the induction hypothesis, Proposition 2.3, and Theorem 2.1(6) in the following estimates

$$\begin{aligned}
|Dh_{n+1}&(y) - Dh_{n+1}(x)| \\
&\leq |D\tilde{\psi}(\tilde{x}_-)(D\hat{h}_n(y_-) - D\hat{h}_n(x_-))D\psi^{-1}(x)| + C \cdot d \cdot |y - x| \\
&\leq \theta_2^4 \cdot C_n \cdot d(R^4 F, R^4 \tilde{F}) \cdot \left(\frac{|y - x|}{\theta_1^4}\right)^\alpha \cdot \frac{1}{\theta_1^4} + C \cdot d \cdot |y - x| \\
&\leq C_n \cdot \left(\frac{\theta_2 \nu}{\theta_1^{1+\alpha}}\right)^4 \cdot d \cdot |y - x|^\alpha + C \cdot d \cdot |y - x|.
\end{aligned}$$

Recall the definition of $\gamma < 1$, see (4.6), and we get

$$C_{n+1} \leq \max\{C_0, \gamma \cdot C_n + C\}.$$

The Lemma follows. $\qquad\square$

Remark 4.3. The proof of Lemma 4.8 and Lemma 4.9 rely on the strong convergence in $W_{\mathrm{loc}}^{ss}(F_*)$. Namely,

$$\frac{\theta_2 \nu}{\theta_1} < 1.$$

Also observe that Lemma 4.7 is only useful under the same condition. The convergence in $W_{\mathrm{loc}}^{s}(F_*)$ is too slow to allow a similar treatment.

The Cantor sets $\mathcal{O}_F$ are constructed similar to the limit set of an iterated function system. See Figure 3.1. The pieces B_{wv}^n are images of the branches Ψ_w^n. These branches are generated by the rescaling maps ψ_0^n and ψ_1^n which have the property $\psi_0^n \to \psi_0$ and $\psi_1^n \to \psi_1$. One could generalize this construction by arbitrarily chosen converging sequences of contracting diffeomorphisms. In this generalized context one would also obtain a rigidity theorem under the condition that the convergence is fast enough compared to the minimal and maximal contraction rates of the limit diffeomorphisms. However, one can construct examples of

this generalized type with slow convergence such that the conjugations are not Lipschitz.

The Rigidity Theorem 4.1 in the context of area-preserving maps relies on the above mentioned general principle together with the fact that the weak stable direction is a family containing the fixed point of analytically conjugated maps, see Theorem 2.1(7).

From Lemma 4.7 we get convergence $\lim Dh_n(x) = D(x)$. The following Lemmas states that the matrices $D(x)$ are indeed the derivatives of the limit map $h = \lim h_n : \mathcal{O}_F \to \mathcal{O}_{\tilde{F}}$ and that these derivatives satisfy a Holder condition. They follow from Lemma 4.8 and Lemma 4.9 by taking the limit.

Lemma 4.10. *For every $0 < \alpha < \alpha_0$ there exists a constant $C > 0$ such that*

$$|h(y) - h(x) - Dh(x)(y - x)| \leq C \cdot d(F, \tilde{F}) \cdot |y - x|^{1+\alpha},$$

where $x, y \in \mathcal{O}_F$ and $Dh(x) = D(x)$.

Lemma 4.11. *For every $0 < \alpha < \alpha_0$ there exists a constant $C > 0$ such that*

$$|Dh(y) - Dh(x)| \leq C \cdot d(F, \tilde{F}) \cdot |y - x|^{\alpha},$$

where $x, y \in \mathcal{O}_F$.

Proof of Propostuion 4.5: Lemma 4.10 and Lemma 4.11 say that the conjugation $h : \mathcal{O}_F \to \mathcal{O}_{\tilde{F}}$ is a $C^{1+\alpha}$-map. By switching the role of F and $\tilde{F}$ we see that h is in fact a $C^{1+\alpha}$-diffeomorphism. The Proposition follows. □

Proof of the Rigidity Theorem 4.1: Given $F \in W^s(F_*)$, let $\tilde{F} \in W^{ss}_{\text{loc}}(F_*)$ and $n \geq 1$ be the map from Lemma 4.4. In particular, the conjugation $h : \mathcal{O}_{R^n F} \to \mathcal{O}_{\tilde{F}}$ is a $C^{1+\text{Lip}}$-diffeomorphism. Proposition 4.5 says,

$$\mathcal{O}_{\tilde{F}} = \mathcal{O}_{F_*} \mod(C^{1+\alpha}).$$

Hence,

$$\mathcal{O}_{R^n F} = \mathcal{O}_{F_*} \mod(C^{1+\alpha}).$$

Apply Lemma 4.2 and the Rigidity Theorem 4.1 follows. □

5. Appendix

Let $D \subset \mathbb{R}^2$ and consider a bounded collection of C^3-maps $\psi : D \to \mathbb{R}^2$. For such a collection there exists a $K > 0$ such that the following Lemmas hold for the error terms

$$E_\psi(x, y) = \psi(y) - \psi(x) - D\psi(x)(y - x).$$

corresponding to affine approximations. We associate these error terms with distortion.

Lemma 5.1. *For every* $x, y, x + \Delta, y + \Delta \in D$

$$|E_\psi(x + \Delta, y + \Delta) - E_\psi(x, y)| \leq K \cdot |\Delta| \cdot |y - x|^2.$$

Lemma 5.2. *For every* $x, y, y + \Delta \in D$

$$|E_\psi(x, y + \Delta) - E_\psi(x, y)| \leq K \cdot \{|\Delta| \cdot |y - x| + |\Delta|^2\}.$$

Lemma 5.3. *For every* $x, y \in D$

$$|E_\psi(x, y)| \leq K \cdot |y - x|^2.$$

Lemma 5.4. *For every* $x, y \in D$

$$|E_{\tilde{\psi}}(x, y) - E_\psi(x, y)| \leq K \cdot ||\tilde{\psi} - \psi||_{C^2} \cdot |y - x|^2.$$

For completeness let us indicate how one can show these estimates. First observe that it suffices to obtain these estimates for the coordinate functions. Secondly, one can reduce the problem to a one-variable problem by restricting these coordinate function to the lines connecting two points. For a one-variable function $\psi : (d_1, d_2) \to \mathbb{R}$ we have

$$E_\psi(x, y) = \int_x^y \psi^{(2)}(t) \cdot (y - t) \cdot dt.$$

One can use this to prove for example Lemma 5.2. Namely,

$$|E_\psi(x, y + \Delta) - E_\psi(x, y)| \leq$$

$$|\int_y^{y+\Delta} \psi^{(2)}(t) \cdot (y - t) \cdot dt + \Delta \cdot \int_x^{y+\Delta} \psi^{(2)}(t) dt|.$$

The Lemma follows.

The next Lemma is responsible for a crucial cancellation in the proof of Lemma 4.8.

Lemma 5.5. *If* $\psi : D \to \psi(D)$ *is a diffeomorphism then*

$$D\psi(\psi^{-1}(x))E_{\psi^{-1}}(x, y) + E_\psi(\psi^{-1}(x), \psi^{-1}(y)) = 0,$$

for every $x, y \in \psi(D)$

The proof is a calculation. Let $x_- = \psi^{-1}(x)$ and $y_- = \psi^{-1}(y)$.

$$
\begin{aligned}
0 &= \psi \circ \psi^{-1}(y) - y \\
&= \psi(\psi^{-1}(x) + D\psi^{-1}(x)(y-x) + E_{\psi^{-1}}(x,y)) - y \\
&= x + D\psi(x_-)\{D\psi^{-1}(x)(y-x) + E_{\psi^{-1}}(x,y)\} + E_\psi(x_-, y_-) - y \\
&= D\psi(\psi^{-1}(x))E_{\psi^{-1}}(x,y) + E_\psi(\psi^{-1}(x), \psi^{-1}(y)).
\end{aligned}
$$

Consider the error terms

$$ ED_\psi(x,y) = D\psi(y) - D\psi(x). $$

Lemma 5.6. *For every* $x, y, x+\Delta, y+\Delta \in D$

$$ |ED_\psi(x+\Delta, y+\Delta) - ED_\psi(x,y)| \le K \cdot |\Delta| \cdot |y-x|. $$

Lemma 5.7. *For every* $x, y, y+\Delta \in D$

$$ |ED_\psi(x, y+\Delta) - ED_\psi(x,y)| \le K \cdot |\Delta|. $$

Lemma 5.8. *For every* $x, y \in D$

$$ |ED_\psi(x,y)| \le K \cdot |y-x|. $$

Lemma 5.9. *For every* $x, y \in D$

$$ |ED_{\tilde\psi}(x,y) - ED_\psi(x,y)| \le K \cdot ||\tilde\psi - \psi||_{C^2} \cdot |y-x|. $$

The next Lemma is responsible for a crucial cancellation in the proof of Lemma 4.9.

Lemma 5.10. *If* $\psi : D \to \psi(D)$ *is a diffeomorphism then*

$$ D\psi(\psi^{-1}(x))ED_{\psi^{-1}}(x,y) + ED_\psi(\psi^{-1}(x), \psi^{-1}(y))D\psi^{-1}(y) = 0, $$

for every $x, y \in \psi(D)$

REFERENCES

[AK]   J. J. Abad, H. Koch, Renormalization and periodic orbits for Hamiltonian flows, Comm. Math. Phys. 212, (2000) # 2, 371–394.

[AKW] J. J. Abad, H. Koch and P. Wittwer, A renormalization group for Hamiltonians: numerical results, Nonlinearity 11 (1998), 1185–1194.

[AL]   A. Avila, M. Lyubich, The full renormalization horseshoe for unimodel maps of higher degree: exponential contraction along hybrid classes, Publ. Math. IHES (2011) 114, Issue 1, 171-223.

[BCGG] G. Benettin et al, Universal properties in conservative dynamical systems, Lettere al Nuovo Cimento 28 (1980) 1-4.

[BMT] C. Birkhoff, M. Martens, C. P. Tresser. On the Scaling Structure for Period Doubling. Asterisque (286) (2003), 167-186.

[Bo]   T. Bountis, Period doubling bifurcations and universality in conservative Systems, Physica 3D (1981), 577-589.

[CEK1] P. Collet, J. P. Eckmann, H. Koch. Period doubling bifurcations for families of maps on $\mathbb{R}^n$. J. Stat. Physics 25 (1980), 1-15.

[CEK2] P. Collet, J.-P. Eckmann and H. Koch, On universality for area-preserving maps of the plane , Physica 3D (1981), 457-467.

[CLM] A. de Carvalho, M. Lyubich, M. Martens. Renormalization in the Hénon family, I: universality but non-rigidity, J. Stat. Phys. 121 No. 5/6, (2005), 611-669.

[CT] P. Coullet, C. Tresser. Itération d'endomorphismes et groupe de renormalisation. J. Phys. Colloque C 539, C5-25 (1978).

[DP] B. Derrida, Y. Pomeau, Feigenbaum's ratios of two dimensional area preserving maps, Phys. Lett. A80 (1980), 217–219.

[EKW1] J.P. Eckmann, H. Koch and P. Wittwer, Existence of a fixed point of the doubling transformation for area-preserving maps of the plane, Phys. Rev. A 26 (1982), # 1 720–722.

[EKW2] J. P. Eckmann, H. Koch, P.Wittwer. A Computer-Assited Proof of Universality for area-preserving Maps, Memoirs of the American Math. Soc. 47 (1984), 1-121.

[F1] M.J. Feigenbaum. Quantitative universality for a class of non-linear transformations. J. Stat. Phys., v. 19 (1978), 25-52.

[F2] M.J. Feigenbaum. The universal metric properties of non-linear transformations. J. Stat. Phys., v. 21 (1979), 669-706.

[FMP] E. de Faria, W. de Melo, A. Pinto. Global hyperbolicity of renormalization for C^r unimodal mappings, Ann. of Math. 164 No. 3, (2006), 731-824.

[G1] D. Gaidashev, Renormalization of isoenergetically degenerate Hamiltonian flows and associated bifurcations of invariant tori, Discrete Contin. Dyn. Syst. 13 (2005), no. 1, 63–102.

[GJ1] D. Gaidashev, T. Johnson, Dynamics of the Universal Area-Preserving Map Associated with Period Doubling: Hyperbolic Sets, Nonlinearity 22 (2009), 2487-2520.

[GJ2] D. Gaidashev, T. Johnson, Hyperbolicity and spectral properties of renormalization for area-preserving maps, in preparation or Institute for Mathematical Sciences SUNY at Stony Brook preprint 2012/04.

[GST] J.-M. Gambaudo, S. van Strien & C. Tresser. Hénon-like maps with strange attractors: there exist C^∞ Kupka-Smale diffeomorphisms on S^2 with neither sinks nor sources. Nonlinearity (1989) 2, 287-304.

[He] M.R. Hermann, Sur la conjugaision differentiable des diffeomorphismes du cercle. Publ. Math. IHES 49 (1976).

[Hl] R. H. G. Helleman, Self-generated chaotic behavior in nonlinear mechanics, in "Fundamental problems in statistical mechanics", Ed. by E. G. D. Cohen, North-Holland, Amsterdam, p.165, (1980).

[K1] H. Koch, On the renormalization of Hamiltonian flows, and critical invariant tori, Discrete Contin. Dyn. Syst. 8 (2002), 633–646.

[K2] H. Koch, A renormalization group fixed point associated with the breakup of golden invariant tori, Discrete Contin. Dyn. Syst. 11 (2004), no. 4, 881–909.

[K3] H. Koch, Existence of critical invariant tori, Ergodic theory and dynamical systems 28 (2008), 1879-94.

[Kh]   K. Khanin, J. Lopes Dias, J. Marklof, Multidimensional continued frac-
        tions, dynamic renormalization and KAM theory, *Comm. Math. Phys.*,
        270 (2007), no. 1, 197–231.

[Ko]   S. Kocić, Renormalization of Hamiltonians for Diophantine frequency vec-
        tors and KAM tori, Nonlinearity 18 (2005), 2513–2544.

[L]   M. Lyubich. Feigenbaum-Coullet-Tresser Universality and Milnor's Hairi-
        ness Conjecture. Ann. Math. v. 149 (1999), 319 - 420.

[LM1]   M. Lyubich, M. Martens. Renormalization of Hénon maps. Dynamics,
        Games and Science I, Springer Proceedings in Mathematics 1, M.M.
        Peixoto, A.A. Pinto, D.A. Rand (Eds.), (2011), 597-618.

[LM2]   M. Lyubich, M. Martens. Renormalization in the Hénon family, II. The
        Heteroclinic Web, Invent. Math. (2001) 186, 115-189.

[LM3]   M. Lyubich, M. Martens. Probabilistic Universality in Two-Dimensional
        Dynamics. IMS Stony Brook preprint 11-2.

[Ma]   M.Martens, *The Periodic Points of Renormalization*, Ann. of Math. 147,
        (1998), 543-584.

[McM]   C. McMullen. Renormalization and 3-manifolds which fiber over the circle.
        Annals of Math. Studies, v. 135, Princeton University Press, 1996.

[Mi]   M.Misiurewicz, Structure of mappings of the interval with zero entropy,
        Publ. Math. IHES (1981), 53, 5-16.

[MS]   W. de Melo, S. van Strien, One-dimensional dynamics, Springer Verlag,
        Berlin, 1993.

[S]   D. Sullivan. Bounds, quadratic differentials, and renormalization conjec-
        tures. AMS Centennial Publications. v. 2: Mathematics into Twenty-first
        Century (1992).

[TC]   C. Tresser, P. Coullet. Itération d'endomorphismes et groupe de renormal-
        isation. C.R. Acad. Sc. Paris v. 287A (1978), 577-580.

[VSK]   E.B. Vul, Ya.G. Sinai, K.M. Khanin. Feigenbaum universality and the ther-
        modynamical formalism. Russian Math, Surveys, v. 39 (1984) , no. 3, 1-40.

[Y]   M. Yampolsky. The attractor of renormalization and rigidity of towers of
        critical circle maps. Comm. Math. Phys. v. 218 (2001), 537-568.

DEPARTMENT OF MATHEMATICS, UPPSALA UNIVERSITY, UPPSALA, SWEDEN,
GAIDASHEV@MATH.UU.SE

FRAUNHOFER-CHALMERS RESEARCH CENTRE FOR INDUSTRIAL MATHEMATICS
CHALMERS UNIVERSITY OF TECHNOLOGY, SE-412 88 GOTHENBURG, SWEDEN,
TOMAS.JOHNSON@FCC.CHALMERS.SE

DEPARTMENT OF MATHEMATICS, STONYBROOK UNIVERSITY, STONY BROOK,
NY11794, USA, MARCO@MATH.SUNYSB.EDU