Period Doubling Renormalization for Area-Preserving Maps and Mild Computer Assistance in Contraction Mapping Principle

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Abstract

A universal period doubling cascade analogous to the famous Feigenbaum-Coullet-Tresser period doubling has been observed in area-preserving maps of $\mathbb{R}^2$. Existence of the "universal" map with orbits of all binary periods has been proved via a renormalization approach in (Eckmann *et al* 1984) and (Gaidashev *et al* 2011). These proofs use "hard" computer assistance.

In this paper we attempt to reduce computer assistance in the argument, and present a mild computer aided proof of the analyticity and compactness of the renormalization operator in a neighborhood of a renormalization fixed point: that is a proof that does not use generalizations of interval arithmetics to functional spaces — but rather relies on interval arithmetics on real numbers only to estimate otherwise explicit expressions. The proof relies on several instance of the Contraction Mapping Principle, which is, again, verified via mild computer assistance.

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1 Introduction: Period doubling universality for area-preserving maps

Following the pioneering discovery of the Feigenbaum-Coullet-Tresser period doubling universality in unimodal maps (Feigenbaum 1978, Feigenbaum 1979, Tresser and Coullet 1978) universality has been demonstrated to be a rather generic phenomenon in dynamics.

To prove universality one usually introduces a *renormalization* operator in a functional space, and demonstrates that this operator has a hyperbolic fixed point.

Such renormalization approach to universality has been very successful in one-dimensional dynamics, and has led to explanation of universality in unimodal maps (Epstein 1986, Epstein 1989, Sullivan 1992, Lyubich 1999), critical circle maps (de Faria 1992, de Faria 1999, Yampolsky 2002, Yampolsky 2003) and holomorphic maps with a Siegel disks (McMullen 1998, Yampolsky 2007, Gaidashev and Yampolsky 2007). Universality has been also abundantly observed in higher dimensions, in particular, in two and more dimensional dissipative systems (cf. (Collet *et al* 1980), (Sparrow 1982)), in area-preserving maps, both as the period doubling universality (Derrida and Pomeau 1980, Helleman 1980, Benettin *et al* 1980, Collet *et al* 1981, Eckmann *et al* 1982, Eckmann *et al* 1984, Gaidashev and Koch 2008, Gaidashev and Johnson 2009, Gaidashev and Johnson 2009, Gaidashev *et al* 2011), and as the universality associated with the break-up of invariant surfaces (Shenker and Kadanoff 1982, MacKay 1982, MacKay 1983, Mehr and Escande 1984), and in Hamiltonian flows (Escande and Doveil 1981, Abad *et al* 2000, Abad *et al* 1998, Koch 2002, Koch 2004, Koch 2008, Gaidashev and Koch 2004, Gaidashev 2005, Kocić 2005, Khanin *et al* 2007).

In this paper we will consider a period doubling universality for area-preserving maps of the plane — an analogue of Feigenbaum-Coullet-Tresser universality in higher dimensions.

An infinite period doubling cascade in families of area-preserving maps was observed by several authors in early 80's (Derrida and Pomeau 1980, Helleman 1980, Benettin *et al* 1980, Bountis 1981, Collet *et al* 1981). A typical period doubling scenario can be illustrated with the area-preserving Hénon family (cf. (Bountis 1981)) :

$$H_a(x, y) = (-y + 1 - ax^2, x).$$

Maps in this family posses a fixed point $((-1 + \sqrt{1+a})/a, (-1 + \sqrt{1+a})/a)$ which is stable for $-1 < a < 3$. When $a_1 = 3$ this fixed point becomes unstable, at the same time an orbit of period two is born with $H_a(x_\pm, x_\mp) = (x_\mp, x_\pm)$, $x_\pm = (1 \pm \sqrt{a-3})/a$. This orbit, in turn, becomes unstable at $a_2 = 4$, giving birth to a period 4 stable orbit. Generally, there exists a sequence of parameter values a_k, at which the orbit of period 2^{k-1} turns unstable, while at the same time a stable orbit of period 2^k is born. The parameter values a_k accumulate on some a_∞. The crucial observation is that the accumulation rate

$$\lim_{k \to \infty} \frac{a_k - a_{k-1}}{a_{k+1} - a_k} = 8.721...$$

is universal for a large class of families, not necessarily Hénon.

Furthermore, the 2^k periodic orbits scale asymptotically with two scaling parameters

$$\lambda = -0.249..., \quad \mu = 0.061...$$

To explain how orbits scale with λ and μ we will follow (Bountis 1981). Consider an interval (a_k, a_{k+1}) of parameter values in a '"typical" family F_a. For any value $\alpha \in (a_k, a_{k+1})$ the map F_α posses a stable periodic orbit of period 2^k. We fix some α_k within the interval (a_k, a_{k+1}) in some consistent way; for instance, by requiring that the restriction of $F_{\alpha_k}^{2^k}$ to a neighborhood of a stable periodic point in the 2^k-periodic orbit is conjugate via a diffeomorphism H_k to a rotation with some fixed rotation number r. Let p'_k be some unstable periodic point in the 2^{k-1}-periodic orbit, and let p_k be the further (from p'_k) of the two stable 2^k-periodic points that bifurcated from p'_k. Then,

$$\frac{1}{\lambda} = -\lim_{k\to\infty} \frac{|p_k - p'_k|}{|p_{k+1} - p'_{k+1}|}, \quad \frac{\lambda}{\mu} = -\lim_{k\to\infty} \frac{\rho_k}{\rho_{k+1}},$$

where ρ_k is the ratio of the eigenvalues of $DH_k(p_k)$.

This universality can be explained rigorously if one shows that the *renormalization* operator

$$R[F] = \Lambda_F^{-1} \circ F \circ F \circ \Lambda_F, \tag{1.1}$$

where Λ_F is some F-dependent coordinate transformation, has a fixed point, and the derivative of this operator is hyperbolic at this fixed point.

It has been argued in (Collet *et al* 1981) that Λ_F is a diagonal linear transformation. Furthermore, such Λ_F has been used in (Eckmann *et al* 1982) and (Eckmann *et al* 1984) in a computer assisted proof of existence of the renormalization fixed point. This is the strongest result to date concerning the existence of the renormalization fixed point.

The problem has been further studied in (Gaidashev and Koch 2008), where the authors have demonstrated that the period doubling renormalization fixed point of (Eckmann *et al* 1984) is "almost" one dimensional, in the sense that it is close to the following Hénon-like map:

$$H_*(x, u) = (\phi(x) - u, x - \phi(\phi(x) - u)),$$

where ϕ solves

$$\phi(x) = \frac{2}{\lambda}\phi(\phi(\lambda x)) - x.$$

(Gaidashev and Koch 2008) also suggests an analytic approach to the full period doubling problem for area-preserving maps based on its proximity to the one dimensional.

Infinitely renormalizable maps, that is maps on the renormalization stable manifold, have been studied in (Gaidashev and Johnson 2009) and (Gaidashev *et al* 2011), where it has been shown that such maps (in a codimension 1 stable submanifold) posses stable Cantor sets (sets on which the Lyapunov exponents

are equal to zero) on which dynamics is rigid: the stable dynamics for two different infinitely renormalizable maps is conjugated by a $C^{1+\alpha}$ transformation (that is, a differentiable transformation whose derivative is Hölder with exponent α). This is in a stark contrast to the situation in dissipative Hénon-like maps, where universality of Cantor sets (around the so called "tip") coexists with non-rigidity of the conjugacy of the dynamics on these sets (see (de Carvalho *et al* 2005, Lyubich and Martens 2008, Hazard *et al* 2010)).

The computer-assisted proof of (Eckmann *et al* 1984) can be described as an application of the Contraction Mapping Principle for a Newton map for the operator (1.1) in some neighborhood in an appropriately chosen Banach space. The proof is similar in spirit to the proofs (Lanford 1982, Lanford 1984) of the Feigenbaum-Coullet-Tresser universality by Oscar Lanford, and uses bounds for analytic functions (not necessarily polynomials) via the so called "standard sets": loosely, generalizations of intervals in $\mathbb{R}$ to infinite-dimensional functional spaces. Such computer assisted proofs came to be called "hard" within the computational community. The approach turned out to be quite powerful, albeit rather specialized and time and effort consuming.

In this paper we study analyticity and compactness of the renormalization operator (1.1) using, almost exclusively, the interval arithmetics in $\mathbb{R}$. We obtain a bound on the neighborhood of analyticity of the operator. Specifically, we identify a low order polynomial approximation for the "generating function" of the fixed point map of (1.1), and estimate the radius of the neighborhood of analyticity of the renormalization operator, centered on this polynomial approximation in an appropriate Banach space of functions analytic on a certain domain in $\mathbb{R}^2$.

Furthermore, we demonstrate that renormalizations of functions from this neighborhood of analyticity, are themselves functions, analytic on a larger domain in $\mathbb{R}^2$, which implies compactness of the renormalization operator.

We achieve these two goals using only what can be termed as "mild" computer assistance: computer estimates on arithmetic operations on reals and on the square root function on reals, as well as norm estimates of low order (quadratic) polynomials. It is also our hope that the reader can find of interest several little "tricks" used in the paper to get bounds on solutions of functional equations.

2 A renormalization operator on generating functions

An "area-preserving map" will mean an exact symplectic diffeomorphism of a subset of $\mathbb{R}^2$ onto its image.

Recall, that an area-preserving map $F : (x, u) \mapsto (y, v)$ can be uniquely specified by its generating function $\mathcal{S}$:

$$\begin{pmatrix} x \\ -\mathcal{S}_1(x, y) \end{pmatrix} \stackrel{F}{\mapsto} \begin{pmatrix} y \\ \mathcal{S}_2(x, y) \end{pmatrix}, \quad \mathcal{S}_i \equiv \partial_i \mathcal{S}, \tag{2.2}$$

if the equation $u = -\mathcal{S}_1(x, y)$ has the unique solution $y(x, u)$.

Furthermore, we will assume that F is reversible, that is

$$T \circ F \circ T = F^{-1}, \quad \text{where} \quad T(x, u) = (x, -u).$$

For such maps it follows from (2.2) that

$$\mathcal{S}_1(y, x) = \mathcal{S}_2(x, y) \equiv s(x, y),$$

and

$$\begin{pmatrix} x \\ -s(y, x) \end{pmatrix} \xmapsto{F} \begin{pmatrix} y \\ s(x, y) \end{pmatrix}.$$

It is this "little" s that will be referred to below as "the generating function". Applying a reversible F twice we get

$$\begin{pmatrix} x' \\ -s(z', x') \end{pmatrix} \xmapsto{F} \begin{pmatrix} z' \\ s(x', z') \end{pmatrix} = \begin{pmatrix} z' \\ -s(y', z') \end{pmatrix} \xmapsto{F} \begin{pmatrix} y' \\ s(z', y') \end{pmatrix}.$$

It has been argued in (Collet *et al* 1981) that

$$\Lambda_F(x, u) = (\lambda x, \mu u),$$

where λ and μ are some real parameters, dependent on F.

We therefore set $(x', y') = (\lambda x, \lambda y)$, $z'(\lambda x, \lambda y) = z(x, y)$ to obtain:

$$\begin{pmatrix} x \\ -\frac{1}{\mu} s(z, \lambda x) \end{pmatrix} \xmapsto{\Lambda} \begin{pmatrix} \lambda x \\ -s(z, \lambda x) \end{pmatrix} \xmapsto{F \circ F} \begin{pmatrix} \lambda y \\ s(z, \lambda y) \end{pmatrix} \xmapsto{\Lambda^{-1}} \begin{pmatrix} y \\ \frac{1}{\mu} s(z, \lambda y) \end{pmatrix}, \qquad (2.3)$$

where $z(x, y)$ solves

$$s(\lambda x, z(x, y)) + s(\lambda y, z(x, y)) = 0. \qquad (2.4)$$

If the solution of (2.4) is unique, then $z(x, y) = z(y, x)$, and it follows from (2.3) that the generating function of the renormalized F is given by

$$\tilde{s}(x, y) = \mu^{-1} s(z(x, y), \lambda y). \qquad (2.5)$$

Furthermore, it is possible to fix some normalization conditions for $\tilde{s}$ and z which serve to determine scalings λ and μ as functions of s. Notice, that the normalization

$$s(1, 0) = 0$$

is reproduced for $\tilde{s}$ as long as

$$z(1, 0) = z(0, 1) = 1.$$

In particular, this implies that

$$s(\lambda, 1) + s(0, 1) = 0.$$

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Furthermore, the condition

$$\partial_1 s(1,0) = 1$$

is reproduced as long as

$$\mu = \partial_1 z(1,0).$$

We will now summarize the above discussion in the following definition of the renormalization operator acting on generating functions originally due to the authors of (Eckmann *et al* 1984):

Definition 2.1

$$
\begin{aligned}
\mathcal{R}_{EKW}[s](x,y) &= \mu^{-1} s(z(x,y), \lambda y), \text{where} \\
0 &= s(\lambda x, z(x,y)) + s(\lambda y, z(x,y)), \\
0 &= s(\lambda, 1) + s(0,1) \quad \text{and} \quad \mu = \partial_1 z(1,0).
\end{aligned}
\tag{2.6}
$$

Definition 2.2 *The Banach space of functions* $s(x,y) = \sum_{i,j=0}^{\infty} c_{ij}(x-\tau)^i (y-\tau)^j$, *analytic on a bi-disk*

$$|x-\tau| < \rho, |y-\tau| < \rho,$$

for which the norm

$$\|s\|_\rho = \sum_{i,j=0}^{\infty} |c_{ij}| \rho^{i+j}$$

is finite, will be referred to as $\mathcal{A}^\tau(\rho)$.

$\mathcal{A}_s^\tau(\rho)$ *will denote its symmetric subspace* $\{s \in \mathcal{A}^\tau(\rho) : s_1(x,y) = s_1(y,x)\}$. *We will also use the shorthand notation*

$$\mathcal{A}(\rho) \equiv \mathcal{A}^0(\rho), \quad \mathcal{A}_s(\rho) \equiv \mathcal{A}_s^0(\rho).$$

As we have already mentioned, the following theorem has been proved with the help of a computer in (Eckmann *et al* 1982) and (Eckmann *et al* 1984), and later in (Gaidashev *et al* 2011). We will quote here the version of the theorem from (Gaidashev *et al* 2011), since we will use the same functional spaces as the proofs in that paper.

Theorem 1 *There exists a polynomial* $s_a : \mathbb{C}^2 \mapsto \mathbb{C}$, *such that*

i) *The operator* $\mathcal{R}_{EKW}$ *is well-defined, analytic and compact in* $\mathcal{B}_r(s_0) \subset \mathcal{A}_s(\rho)$, *with*

$$\rho = 1.75, \quad r = 1.1 \times 10^{-10}.$$

ii) *There exists a function* $s^* \in \mathcal{B}_r(s_0) \subset \mathcal{A}_s(\rho)$ *such that*

$$\mathcal{R}_{EKW}[s^*] = s^*.$$

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iii) The linear operator $D\mathcal{R}_{EKW}[s^*]$ has two eigenvalues outside of the unit circle:

$$8.72021484375 \leq \delta_1 \leq 8.72216796875, \quad \delta_2 = \frac{1}{\lambda_*},$$

where

$$-0.248875313689 \leq \lambda_* \leq -0.248886108398438.$$

iv) The complement of these two eigenvalues in the spectrum is compactly contained in the unit disk. The largest eigenvalue in the unit disk is equal to λ_*, while

$$\mathrm{spec}(D\mathcal{R}_{EKW}[s^*]) \setminus \{\delta_1, \delta_2, \lambda_*\} \subset \{z \in \mathbb{C} : |z| \leq 0.1258544921875\}.$$

In this paper we will adopt a normalization condition different from (2.6) however. Specifically, μ will be defined from the normalization condition

$$\mathcal{R}_{EKW}[s](0,0) = 1,$$

i.e

$$\mu[s] = s(z(0,0),0). \tag{2.7}$$

Definition 2.3 *Define $\mathcal{I}^\tau(\rho)$ and $\mathcal{I}_s^\tau(\rho)$ to be the subsets of $\mathcal{A}^\tau(\rho)$ and $\mathcal{A}^\tau(\rho)$, respectively, of functions normalized in the following way*

$$
\begin{aligned}
\mathcal{I}^\tau(\rho) &= \{s \in \mathcal{A}^\tau(\rho) : s(0,0) = 1\}, \\
\mathcal{I}_s^\tau(\rho) &= \{s \in \mathcal{A}_s^\tau(\rho) : s(0,0) = 1\}, \\
\mathcal{I}(\rho) &= \{s \in \mathcal{A}(\rho) : s(0,0) = 1\}, \\
\mathcal{I}_s(\rho) &= \{s \in \mathcal{A}_s(\rho) : s(0,0) = 1\}.
\end{aligned}
$$

Our main result concerning the operator 2.1 will be the following

Main Theorem 1 *There exists a polynomial $s_0 \in \mathcal{I}(\rho)$, $\rho = 1.75$, of degree 9 such that*

i) the operator (2.1) is analytic in $B_\delta(s_0) \subset \mathcal{A}(\rho)$ with $\delta = 0.00405550003051758$;

ii) for all $s \in B_\delta(s_0)$ with real Taylor coefficients, the scalings $\lambda = \lambda[s]$ and $\mu = \mu[s]$ satisfy

$$
\begin{aligned}
0.000406771898269653 &\leq \mu &\leq 0.120654106140137, \\
-0.276069164276123 &\leq \lambda &\leq -0.222213745117188;
\end{aligned}
$$

iii) the operator (2.1) is compact, with $\mathcal{R}_{EKW}[s] \in \mathcal{I}(\rho')$, $\rho' = 1.0699462890625\rho$, for all $s \in B_\delta(s_0)$.

Remark 2.4 *We would like to note that the renormalization fixed point s^* satisfies*

$$\|s^* - s_0\|_\rho \leq .00368565320968628,$$

(which is a rigorous bound), i.e., as expected, according to Theorem 1, lies in the analyticity domain of renormalization.

Remark 2.5 *All the numbers quoted in the Theorem and the Remark above are representable on a computer.*

Although analyticity and compactness of the renormalization operator have been already proved in (Eckmann *et al* 1984) (together with a stronger result of existence of the fixed point), we would like to reiterate that the goal of the paper will be to obtain these results with a lighter machinery than that of (Eckmann *et al* 1984). We also obtain a better bound on the size of the neighborhood of analyticity of (2.1).

3 Interval arithmetics in $\mathbb{R}$ and $\mathbb{R}^2$

We will now give a very brief summary of interval operations in $\mathbb{R}$ and $\mathbb{R}^2$. For a more complete treatise of "standard sets" and operations on them, an interested reader is referred to an excellent review (Koch *et al* 1996).

A computer implementation of an arithmetic operation $r_1 \# r_2$ (# is $+$, $-$, $*$ or $/$) on two real numbers does not generally yield an exact result. The "computer" result is a number representable in a standard IEEE floating point format (cf (IEEE 1985)). Such numbers are commonly referred to as "representable". In the 80-bit extended precision IEEE arithmetics, a number is represented with 80 bits of memory: 1 bit for the sign of the number, 15 bits for an exponents, and 64 bits for a mantissa. A real representable number r is of a the form

$$r = \pm 1.m \; 2^{e-(2^{14}-1)},$$

where $\pm$ is chosen according to whether the sign bit is 0 or 1, m is a base 2 mantissa given by the sequence of 0's and 1's associated with the 64 bits of the mantissa, and the exponent $0 \leq e \leq 32767$ is defined by the state of the 15 exponent bits (the state of 15 ones is reserved to represent "overflows"). The representable number 0 is given by the sequence of 80 zeros. The set of all representable numbers will be denoted $\mathcal{R}$.

Now, let $r_1 \# r_2$ be a mathematically legal, nonzero, arithmetic operation. The result of the true arithmetic operation $r_1 \# r_2$ might not be a representable number. However, we can instruct the computer to attempt to round this operation either to the nearest representable number, up or down (our choice was to always round up). This might not be possible: the result of rounding up might have the exponent $e \geq 32767$ (overflow), or $e < 0$ (underflow). In both cases the computer is instructed to raise an exception, which is appropriately handled (either by terminating the program, or restarting with a different set of parameters). If, however, the result of rounding up is representable, the output of the

computer implementation of the arithmetic operation is an upper bound on the true result of the operation. We will refer to such bound as "the upper bound".

We will define the *standard sets* in $\mathbb{R}$ to be the collection of all closed real intervals $I[x, y] = \{r \in \mathbb{R} | x \leq r \leq y\}$:

$$\text{std}(\mathbb{R}) = \{I[x, y] \in \mathbb{R} : x, y \in \mathcal{R}\}. \qquad (3.8)$$

If $I[x, y]$, $I[x_1, y_1]$ and $I[x_2, y_2]$ are in $\text{std}(\mathbb{R})$ then we can use the rounding up described above to obtain bounds on the arithmetic operation on these sets as follows:

1) *unary minus*: $-(I[x, y]) = I[-y, -x]$;

2) *absolute value*: $|I[x, y]| = I[l, r]$, where $l = \max\{0, x, -y\}$, $r = -\min\{0, x, -y\}$;

3) *addition*: $I[x_1, y_1] + I[x_2, y_2] = I[x_3, y_3]$, where y_3 is the upper bound on $y_1 + y_2$, while $-x_3$ is the upper bound on $-x_1 + (-x_2)$;

4) *subtraction*: $I[x_1, y_1] - I[x_2, y_2] \equiv I[x_1, y_1] + (-I[x_2, y_2])$;

5) *multiplication*: $I[x_1, y_1] \cdot I[x_2, y_2] = I[x_3, y_3]$, where y_3 is the maximum of the upper bounds on $x_1 \cdot x_2$, $x_1 \cdot y_2$, $y_1 \cdot x_2$ and $y_1 \cdot y_2$, while $-x_3$ is the maximum of the upper bounds on $(-x_1) \cdot x_2$, $(-x_1) \cdot y_2$, $(-y_1) \cdot x_2$ and $(-y_1) \cdot y_2$;

6) *inverse*: if $x_1 \cdot y_1 > 0$, then $I[x_1, y_1]$ inverse $= I[x_2, y_2]$, where y_2 is the upper bound on $1/x_1$ and $-x_2$ is the upper bound on $1/(-y_1)$;

7) *division*: if $x_2 \cdot y_2 > 0$, then $I[x_1, y_1]/I[x_2, y_2] \equiv I[x_1, y_1] \cdot (I[x_2, y_2]$ inverse$)$.

A standard set in $\mathbb{R}^2$ is, naturally, a direct product of two standard sets in $\mathbb{R}$:

$$\text{std}(\mathbb{R}^2) = \{I[x_1, y_1] \times I[x_2, y_2], x_1, x_2, y_1, y_2 \in \mathcal{R}\}. \qquad (3.9)$$

The arithmetic operation on these standard sets in $\mathbb{R}^2$ are reducible to those on reals in an obvious way.

To obtain a bound on a algebraic and transcendental function, one can use their Taylor series together with a bound on the remainder. In our proofs we will require only three such functions: exp, ln and $\sqrt{}$. All of them are implemented via a finite truncation of their Taylor series with a bound on the remainder.

4 Contraction Mapping Principle

We will now outline a rather general method for finding a fixed point of a hyperbolic operator in a Banach space via its approximate Newton map.

Let C be an operator analytic and hyperbolic on some neighborhood $\mathcal{N}$ in a Banach space $\mathcal{Z}$. Suppose, that one knows its approximate hyperbolic fixed point $Z_0 \in \mathcal{N}$. Set

$$M \equiv [\mathbb{I} - DC[Z_0]]^{-1},$$

and for all z, such that $Z_0 + Mz \in \mathcal{N}$,

$$N[z] = z + C[Z_0 + Mz] - (Z_0 + Mz).$$

Notice, that if z^* is a fixed point of N, then $Z_0 + Mz^*$ is a fixed point of C.

The linear operator $\mathbb{I} - DC[Z_0]$ is indeed invertible since DC is hyperbolic at Z_0. If Z_0 is a reasonably good approximation of the true fixed point of C, then the operator N is expected to be a strong contraction in a neighborhood of 0:

$$
\begin{aligned}
DN[z] &= \mathbb{I} + DC[Z_0 + Mz] \cdot M - M \\
&= \left[M^{-1} + DC[Z_0 + Mz] - \mathbb{I} \right] \cdot M \\
&= \left[\mathbb{I} - DC[Z_0] + DC[Z_0 + Mz] - \mathbb{I} \right] \cdot M \\
&= \left[DC[Z_0 + Mz] - DC[Z_0] \right] \cdot M.
\end{aligned}
$$

The last expression is typically small in a small neighborhood of 0, if the norm of M is not too large (if M is large, one might have to find a better approximation Z_0 and take a smaller neighborhood of 0). The following well-known Theorem specifies a sufficient condition for existence of the fixed point:

Theorem 4.1 *(Contraction Mapping Principle)*

Suppose that the operator N is well-defined and analytic as a map from $\mathcal{N} \subset \mathcal{Z}$ to $\mathcal{Z}$, where $\mathcal{Z}$ is some Banach space. Let $Z_0 \in \mathcal{N}$ and $B_\delta(Z_0) \subset \mathcal{N}$ (an open ball of radius δ around Z_0) be such that

$$\|DN[Z]\| \leq \mathcal{D} < 1,$$

for any $Z \in B_\delta(Z_0)$, and

$$\|N[Z_0] - Z_0\| \leq \epsilon.$$

If $\epsilon < (1 - \mathcal{D})\delta$ then the operator N has a fixed point Z^* in $B_\delta(Z_0)$, such that

$$\|Z^* - Z_0\| \leq \frac{\epsilon}{1 - \mathcal{D}}.$$

We will use the above Contraction Mapping Principle in several instances in our proofs below. In all those cases we will be verifying the hypothesis of the Contraction Mapping Principle using mild computer assistance.

5 Analyticity of renormalization

We consider the space $\mathcal{A}(\rho)$, $\rho = 1.75$. Define

$$s_0(x, y) \equiv D_0(x)y^3 + A_0(x)y^2 + B_0(x)y + C_0(x),$$

a polynomial of degree 3, with

$$D_0(x) = \sum_{i=0}^{6} d_i x^i, \quad A_0(x) = \sum_{i=0}^{6} a_i x^i, \quad B_0(x) = \sum_{i=0}^{6} b_i x^i, \quad C_0(x) = \sum_{i=0}^{6} c_i x^i,$$

where the coefficients of these polynomials are as follows (the numbers given in the table are highly accurate approximations of the representable numbers actually used in the programs):

i	c_i	b_i
0	1.00000000000000000	$-2.42962369607899157 \times 10^{-1}$
1	-1.02761956458970711	$5.87327440047455615 \times 10^{-2}$
2	$2.93663720023727808 \times 10^{-2}$	$-5.93710236103475834 \times 10^{-3}$
3	$-1.87658664952086400 \times 10^{-3}$	$6.09332694202817819 \times 10^{-4}$
4	$1.40668294317213841 \times 10^{-4}$	$-6.46957663100331420 \times 10^{-5}$
5	$-1.18664608613747513 \times 10^{-5}$	$7.02844653606969302 \times 10^{-6}$
6	$1.06935654680404746 \times 10^{-6}$	$-7.75814237637266867 \times 10^{-7}$

i	a_i	d_i
0	$-8.77647505670140721 \times 10^{-1}$	$-1.46791670728014469 \times 10^{-3}$
1	$-5.62975994856259201 \times 10^{-3}$	$5.62673177268855366 \times 10^{-4}$
2	$9.13999041304226728 \times 10^{-4}$	$-1.29391532620066284 \times 10^{-4}$
3	$-1.33251463600020727 \times 10^{-4}$	$2.45371180652649971 \times 10^{-5}$
4	$1.84028385489487478 \times 10^{-5}$	$-4.18641657324405651 \times 10^{-6}$
5	$-2.46417657757601024 \times 10^{-6}$	$6.68319468409332288 \times 10^{-7}$
6	$3.23459112039180563 \times 10^{-7}$	$-1.01939399249366523 \times 10^{-7}$.

A numerical study demonstrates that s_0 is a relatively good approximation of the renormalization fixed point in $\mathcal{A}_s(\rho)$ for a rather wide range of radii ρ ($\rho \approx 0.8 \ldots 2.5$, below ρ will be fixed to be 1.75).

In this Section we will prove Theorem 1. This Theorem is proved with a mild aid of the computer. The computer assistance is restricted only to interval arithmetics on real numbers, (which, given sufficient time, in principle, can be done "by hand").

We will start by noticing that a solution of the equation

$$s(x, Z') + s(y, Z') = 0$$

with $s \equiv s_0$ is given by

$$Z_0'(x, y) = \frac{B_0(x) + B_0(y)}{2(A_0(x) + A_0(y))} \left[\sqrt{1 - 4\frac{(A_0(x) + A_0(y))(\tilde{C}_0(x) + \tilde{C}_0(y))}{(B_0(x) + B_0(y))^2}} - 1 \right],$$

where $\tilde{C}_0(x)$ is a bound on $C_0(x) + y^3 D(x)$, specifically:

$$\tilde{C}_0(x) = C_0(x) + C D_0(x), \quad C \in \mathbb{D}_{\rho^3}(0). \tag{5.10}$$

In the formula above we choose a branch of $\sqrt{}$ such that the range of Z_0' is expected to be contained in D_ρ.

We would like to remark that we have intentionally chosen a polynomial s_0 of a sufficiently high degree so that it would be close to the fixed point s^* found in (Gaidashev *et al* 2011). This will later imply that the fixed point s^* is indeed in the domain of analyticity and compactness of $\mathcal{R}_{EKW}$. At the same time, taking a polynomial of degree higher than 3 in y is superfluous for the purposes of the

computation of Z'. Therefore, these extra degrees (polynomial D_0) have been included as an "error term" in (5.10). It is clear, that

$$C_0(x) + y^3 D_0(x) \in \tilde{C}_0(x)$$

for all (x, y) such that $|x| < \rho, |y| < \rho$.

We shall now check that for the given values of coefficients the function Z'_0 is analytic on a bi-disk of radius $r = 0.483119964599609$ centered at point $(0, 0)$. Below $\mathcal{Z}(r)$ will denote the Banach space of functions analytic on $\{(x, y) : |x| < r, |y| < r\}$ equipped with the sup-norm, denoted by $|\cdot|_r$. We have the following

Lemma 5.1 Z'_0 is in $\mathcal{Z}(r)$ for $r = 0.483119964599609$, and satisfies

$$|Z'_0|_r < 1.562789916992188.$$

Proof. First, define constants v_0, β_0 and γ_0, and quartic polynomials v, β and γ, such that $v(0,0) = \beta(0,0) = \gamma(0,0) = 0$ by setting

$$
\begin{aligned}
v_0 + v(x, y) &= A_0(x) + A_0(y), \\
\beta_0 + \beta(x, y) &= B_0(x) + B_0(y), \\
\gamma_0 + \gamma(x, y) &= \tilde{C}_0(x) + \tilde{C}_0(y).
\end{aligned}
$$

Then

$$Z'_0(x, y) = \frac{\beta_0 + \beta(x, y)}{2(v_0 + v(x, y))} \left[\sqrt{1 - 4\frac{v_0 \gamma_0}{\beta_0^2} + F_1(x, y) - F_2(x, y)} - 1 \right], \quad (5.11)$$

where

$$
\begin{aligned}
F_1(x, y) &= \frac{4 v_0 \gamma_0}{\beta_0^2} \frac{2\beta_0 \beta(x, y) + \beta^2(x, y)}{\beta_0^2 + 2\beta_0 \beta(x, y) + \beta^2(x, y)}, \\
F_2(X, Y) &= 4 \frac{v_0 \gamma(x, y) + v(x, y) \gamma_0 + v(x, y) \gamma(x, y)}{\beta_0^2 + 2\beta_0 \beta(x, y) + \beta^2(x, y)}.
\end{aligned}
$$

Norms of both F_1 and F_2 are elementary:

$$|F_1|_r \leq \frac{4|v_0 \gamma_0|}{\beta_0^2} \frac{|2\beta_0 \beta + \beta^2|_r}{\beta_0^2 - |2\beta_0 \beta + \beta^2|_r}, \quad |F_2|_r \leq 4 \frac{|v_0 \gamma + v\gamma_0 + v\gamma|_r}{\beta_0^2 - |2\beta_0 \beta + \beta|_r^2}.$$

Notice that all functions here whose norm has to be evaluated are explicit polynomials. We have estimated the above norms using the interval arithmetics on a computer. Next, we use these bounds in the following expression

$$|Z'_0|_r \leq \frac{|\beta_0| + |\beta|_r}{2(|v_0| - |v|_r)} \max_{j=\pm 1} \left\{ \sqrt{1 - 4\frac{v_0 \gamma_0}{\beta_0^2} + j|F_1|_r + j|F_2|_r} - 1 \right\},$$

and evaluate it on a computer.

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□

Below, we will use the following shorthand notation for the supremum of Z_0':

$$t \equiv |Z_0'|_r. \tag{5.12}$$

Notice that if one writes

$$s(x,y) \equiv \sigma(x,y) - \frac{\tau(y)}{2},$$

where τ is chosen so that it has an inverse branch ν with a range in D_ρ (it is sufficient to choose τ a quadratic function), then a solution of equation (5) satisfies

$$Z'(x,y) = \nu[\sigma(x, Z'(x,y)) + \sigma(y, Z'(x,y))].$$

(this "trick" has been shown to me by Hans Koch).

We will now specify the choices of τ and σ for s_0. Set

$$
\begin{aligned}
s_0(x,y) &\equiv \sigma_0(x,y) - \frac{\tau(y)}{2}, \\
\sigma_0(x,y) &\equiv a(x)y^2 + b(x)y + c(x), \quad \text{and} \quad \Sigma_0(x,y) = \sigma_0(x+p, y), \\
\tau(y) &\equiv -2a_0 y^2 - 2b_0 y, \\
\nu(x) &\equiv \frac{-b_0 + \sqrt{b_0^2 - 2a_0 x}}{2a_0}.
\end{aligned}
$$

It can be readily verified that $Z_0'(x,y) = \nu[\Sigma_0(x, Z_0'(x,y)) + \Sigma_0(y, Z_0'(x,y))]$. We will next demonstrate that the operator C_h, defined on $\mathcal{Z}(r)$ by setting

$$C_h[Z'] \equiv \nu\left[\Sigma_0 \circ (\Pi_1, Z') + h \circ (P_1, Z') + \Sigma_0 \circ (\Pi_2, Z') + h \circ (P_2, Z')\right],$$

where

$$P_1(x,y) \equiv x + p, \quad \text{and} \quad P_2(x,y) \equiv y + p,$$

has a fixed point for sufficiently small h's.

Proposition 5.2 *Let δ, ρ and r be as in Theorem 1 and Lemma 5.1. Then, for all $s \in B_\delta(s_0) \subset \mathcal{A}(\rho)$ the operator C_{s-s_0} has a unique fixed point Z_s' in $B_\epsilon(Z_0') \subset \mathcal{Z}(r)$ with $\epsilon = 0.01465$, and the map $s \mapsto Z_s'$ is analytic from $B_\delta(s_0)$ to $B_\epsilon(Z_0')$.*

Proof. Define an operator

$$N_h[z] = z + C_h[Z_0' + Mz] - (Z_0' + Mz), \quad M \equiv \left[\mathbb{I} - DC_0[Z_0']\right]^{-1}.$$

We will demonstrate that this "Newton map" has a fixed point in a neighborhood of 0.

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We will first estimate the norm of $N_h[0] = C_h[Z_0'] - Z_0'$:

$$
\begin{aligned}
|C_h[Z_0'] - Z_0'|_r &= \left| \nu \left[\Sigma_0 \circ (\Pi_1, Z_0') + h \circ (\Pi_1, Z_0') + \Sigma_0 \circ (\Pi_2, Z_0') + h \circ (\Pi_2, Z_0') \right] \right. \\
&\qquad \left. - \nu \left[\Sigma_0(\Pi_1, Z_0') + \Sigma_0(\Pi_2, Z_0') \right] \right|_r \\
&= \frac{1}{2a_0} \left| \sqrt{b_0^2 - 2a_0 \left[\Sigma_0(\Pi_1, Z_0') + h(P_1, Z_0') + \Sigma_0(\Pi_2, Z_0') + h(P_2, Z_0') \right]} \right. \\
&\qquad \left. - \sqrt{b_0^2 - 2a_0 \left[\Sigma_0(\Pi_1, Z_0') + \Sigma_0(\Pi_2, Z_0') \right]} \right|_r \\
&= \frac{1}{2a_0} |F(1) - F(0)|_r \,,
\end{aligned}
$$

where

$$
F(\zeta) = \sqrt{b_0^2 - 2a_0 \left[\Sigma_0(\Pi_1, Z_0') + \zeta h(P_1, Z_0') + \Sigma_0(\Pi_2, Z_0') + \zeta h(P_2, Z_0') \right]}
$$

is an analytic function from $\mathbb{D}_R$ to $\mathcal{Z}(r)$ (here and below, $\mathbb{D}_R$ stands for an open disk of radius R in $\mathbb{C}$), with

$$
R \leq \frac{b_0^2 - 4a_0\theta_0 - 2|a_0||\Theta_0(\Pi_1, Z_0')|_r - 2|a_0||\Theta_0(\Pi_2, Z_0')|_r}{2|a_0||h(P_1, Z_0')|_r + 2|a_0||h(P_2, Z_0')|_r}, \tag{5.13}
$$

where

$$
\theta_0 \equiv \Sigma_0(0,0), \quad \Theta_0(x,y) \equiv \Sigma_0(x,y) - \theta_0.
$$

Below, we will denote the (positive) real number $b_0^2 - 4a_0\theta_0$ by c. A straightforward Cauchy estimate yields:

$$
|F(1) - F(0)| \leq \frac{1}{R-1} \sup_{|\zeta| \leq R} |F(\zeta)| \leq \frac{4|a_0|\delta\sqrt{2c}}{c - 4|a_0||\Theta_0|_{(r,t)} - 4|a_0|\delta}.
$$

We will denote

$$
\varepsilon \equiv \frac{2\delta\sqrt{2c}}{c - 4|a_0||\Theta_0|_{(r,t)} - 4|a_0|\delta}. \tag{5.14}
$$

This expression has been shown to satisfy

$$
\varepsilon < 0.0137615203857422.
$$

As a next step we will provide a bound on the derivative of the operator N_h:

$$
\begin{aligned}
|DN_h[z]| &= \left| \left[DC_h[Z_0' + Mz] - DC_0[Z_0'] \right] M \right| \\
&\leq \left| \left[DC_h[Z_0' + Mz] - DC_0[Z_0'] \right] \right| |M|.
\end{aligned}
$$

To estimate the norm of M we first verify that $|DC_0[Z_0']| < 1$:

$$
\begin{aligned}
|DC_0[Z_0']| &= \left| \frac{\partial_2\Theta_0(\Pi_1, Z_0') + \partial_2\Theta_0(\Pi_2, Z_0')}{2\sqrt{c - 2a \left[\Theta_0(\Pi_1, Z_0') + \Theta_0(\Pi_2, Z_0') \right]}} \right| \\
&\leq \frac{|\partial_2\Theta_0|_{(r,t)}}{\sqrt{c - 4|a_0| |\Theta_0|_{(r,t)}}}.
\end{aligned}
$$

A computer-aided evaluation of the last expression indeed shows that it is less than 1. This implies that M can be found as the limit of a convergent series whose norm is easily bounded:

$$|M| \leq \frac{1}{1 - |DC_0[Z_0']|} < 1.0430755615234375 \equiv \mathcal{M}. \qquad (5.15)$$

Next,

$$
\begin{aligned}
DC_h[Z'] - DC_0[Z_0'] &= \frac{\partial_2\Theta_0(\Pi_1, Z') + \partial_2\Theta_0(\Pi_2, Z') + \partial_2 h(P_1, Z') + \partial_2 h(P_2, Z')}{2\sqrt{c - 2a_0\left[\Theta_0(\Pi_1, Z') + \Theta_0(\Pi_2, Z') + h(P_1, Z') + h(P_2, Z')\right]}} \\
&\quad - \frac{\partial_2\Theta_0(\Pi_1, Z_0') + \partial_2\Theta_0(\Pi_2, Z_0')}{2\sqrt{c - 2a_0\left[\Theta_0(\Pi_1, Z_0') + \Theta_0(\Pi_2, Z_0')\right]}} \\
&= \left[\frac{\partial_2\Theta_0(\Pi_1, Z') + \partial_2\Theta_0(\Pi_2, Z') + \partial_2 h(P_1, Z') + \partial_2 h(P_2, Z')}{2\sqrt{c - 2a_0\left[\Theta_0(\Pi_1, Z') + \Theta_0(\Pi_2, Z') + h(P_1, Z') + h(P_2, Z')\right]}}\right. \\
&\quad \left. - \frac{\partial_2\Theta_0(\Pi_1, Z') + \partial_2\Theta_0(\Pi_2, Z')}{2\sqrt{c - 2a_0\left[\Theta_0(\Pi_1, Z') + \Theta_0(\Pi_2, Z') + h(P_1, Z') + h(P_2, Z')\right]}}\right] \\
&\quad + \left[\frac{\partial_2\Theta_0(\Pi_1, Z') + \partial_2\Theta_0(\Pi_2, Z')}{2\sqrt{c - 2a_0\left[\Theta_0(\Pi_1, Z') + \Theta_0(\Pi_2, Z') + h(P_1, Z') + h(P_2, Z')\right]}}\right. \\
&\quad \left. - \frac{\partial_2\Theta_0(\Pi_1, Z') + \partial_2\Theta_0(\Pi_2, Z')}{2\sqrt{c - 2a_0\left[\Theta_0(\Pi_1, Z') + \Theta_0(\Pi_2, Z')\right]}}\right] \\
&\quad + \left[\frac{\partial_2\Theta_0(\Pi_1, Z') + \partial_2\Theta_0(\Pi_2, Z')}{2\sqrt{c - 2a_0\left[\Theta_0(\Pi_1, Z') + \Theta_0(\Pi_2, Z')\right]}}\right. \\
&\quad \left. - \frac{\partial_2\Theta_0(\Pi_1, Z_0') + \partial_2\Theta_0(\Pi_2, Z_0')}{2\sqrt{c - 2a_0\left[\Theta_0(\Pi_1, Z_0') + \Theta_0(\Pi_2, Z_0')\right]}}\right] \\
&\equiv I_1 + I_2 + I_3. \qquad (5.16)
\end{aligned}
$$

We will estimate norms of the three expressions in brackets separately.

$$|I_1|_r \leq \frac{|\partial_2 h|_{(r,s)}}{\sqrt{c - 4|a_0||\Theta_0|_{(r,s)} - 4|a_0||h|_{(r,s)}}} \leq \frac{m\delta}{\sqrt{c - 4|a_0||\Theta_0|_{(r,s)} - 4|a_0|\delta}}, \quad (5.17)$$

where

$$s = t + |M|\epsilon, \quad m = n\frac{s^{n-1}}{\rho^n} \quad \text{and} \quad n = \frac{1}{\ln\left[\frac{\rho}{s}\right]}.$$

We use a Cauchy estimate for the function

$$F_2(\zeta) = \frac{1}{\sqrt{c - 2a_0\left[\Theta_0(\Pi_1, Z') + \Theta_0(\Pi_2, Z') + \zeta h(P_1, Z') + \zeta h(P_2, Z')\right]}}$$

to bound $|I_2|_r$. Notice, that, F_2 is also analytic on $\mathbb{D}_R$, but gets unbounded as ζ approaches the boundary of this disk. Therefore, to bound $|I_2|_r$, we use a Cauchy bound for F_2 on a smaller disk of radius

$$\tilde{R} = \frac{c - 4|a_0| \, |\Theta_0|_{(r,s)}}{4|a_0|\delta} - \frac{1}{3} \frac{c - 4|a_0| \, |\Theta_0|_{(r,s)} - 4|a_0|\delta}{4|a_0|\delta} \; :$$

$$|I_2|_r \le |\partial_2 \Theta_0|_{(r,s)} \frac{1}{\tilde{R} - 1} \sup_{|\zeta| \le \tilde{R}} |F_2(\zeta)| \le \frac{4|a_0|\delta \, |\partial_2 \Theta_0|_{(r,s)}}{2 \left(\frac{1}{3} \left(c - 4|a_0| \, |\Theta_0|_{(r,s)} - 4|a_0|\delta \right) \right)^{3/2}}.$$

Finally, to estimate $|I_3|_r$, we use a Cauchy bound for the function

$$F_3(\zeta) = \frac{\partial_2 \Theta_0(\Pi_1, Z_0' + \zeta M z) + \partial_2 \Theta_0(\Pi_2, Z_0' + \zeta M z)}{2\sqrt{c - 2a_0 \left[\Theta_0(\Pi_1, Z_0' + \zeta M z) + \Theta_0(\Pi_2, Z_0' + \zeta M z) \right]}},$$

analytic on $\mathbb{D}_{\hat{R}}$ with

$$\hat{R} = \frac{\rho - t}{|M|\epsilon} \; :$$

$$|I_3|_r = |F_3(1) - F_3(0)| \le \frac{1}{\hat{R} - 1} \sup_{|\zeta| \le \hat{R}} |F_3(\zeta)| \le \frac{|M|\epsilon}{\rho - s} \frac{|\partial_2 \Theta_0|_{(r,\rho)}}{\sqrt{c - 4|a_0||\Theta_0|_{(r,\rho)}}}. \quad (5.18)$$

Individual norms $|M|$, $|I_1|_r$, $|I_2|_r$ and $|I_3|_r$ and their sum have been estimated on a computer to produce:

$$|DN_h(z)| \le 0.0125999450683594 \equiv \mathcal{D} \quad (5.19)$$

for all $\|h\|_\rho < \delta$ and $|z|_r < \epsilon$.

Estimates (5.14) and (5.19) demonstrate that

$$\varepsilon < (1 - \mathcal{D})\frac{\epsilon}{\mathcal{M}},$$

and the claim follows from the Contraction Mapping Principle.

$\square$

At the next step we obtain bounds on the scaling parameters λ and μ.

Recall that the scaling λ solves equation (2.6).

The prove Theorem 1 we will require the following

Lemma 5.3 *Equation* (2.6) *has a solution* $\lambda = \lambda[s]$ *for all* $s \in B_\delta(s_0)$, *and satisfies*

$$-0.276069164276123 \le \lambda \le -0.222213745117188$$

for all $s \in B_\delta(s_0)$ *with real Taylor coefficients.*

Furthermore, the map $s \mapsto \lambda[s]$ *is analytic on* $B_\delta(s_0)$.

Proof. Write,

$$s_1(x,y) \equiv A_1(y)x^2 + B_1(y)x + C_1(y),$$

where

$$
\begin{aligned}
A_1(y) &= c_2 + b_2 y + a_2 y^2 + d_2 y^3, \\
B_1(y) &= c_1 + b_1 y + a_1 y^2 + d_1 y^3, \\
C_1(y) &= c_0 + b_0 y + a_0 y^2 + d_0 y^3.
\end{aligned}
$$

Denote

Notice, that the equation (2.6) with $s = s_1$ can be solved exactly (a quadratic equation for λ). Furthermore, suppose $s = s_1 + h$, $\|h\|_\rho \leq \|s_0 - s_1\|_\rho + \delta \equiv \tilde{\delta}$, then $\lambda[s]$ solves

$$
s_1(\lambda[s], 1) + s_1(0, 1) = -h(\lambda[s], 1) - h(0, 1). \tag{5.20}
$$

Write

$$
\lambda[s] = \lambda[s_1] + \Delta\lambda,
$$

then $\Delta\lambda$ is a solution of

$$
A_1(1) \cdot (\Delta\lambda)^2 + [2A_1(1) \cdot \lambda[s_1] + B_1(1)] \cdot \Delta\lambda + C = 0 \tag{5.21}
$$

with $C \in \mathbb{D}_{2\tilde{\delta}}$ (an open disk of radius $2\tilde{\delta}$ in $\mathbb{C}$). For all s with real coefficients, (5.21) is a quadratic equation for $\Delta\lambda$ with real coefficients, whose solution gives the required bound.

□

The last ingredient in the proof of analyticity of renormalization is a bound on the scaling μ.

Lemma 5.4 *Let $\mu[s]$ be as in equation (2.7). Then the map $s \mapsto \mu[s]$ is analytic on $B_\delta(s_0)$, and $\mu[s]$ satisfies*

$$
0.000406771898269653 \leq \mu \leq 0.120654106140137,
$$

for $s \in B_\delta(s_0)$ with real Taylor coefficients.

Proof. The bound is straightforward:

$$
\begin{aligned}
\mu[s] &= s(Z_s'(0,0), 0) \\
&\in s_0(Z_s'(0,0), 0) + h(Z_s'(0,0), 0).
\end{aligned}
$$

Now, it is clear, that whenever s has real coefficients,

$$
\mu[s] \in s_0(Z_0'(0,0) + I[-|M|\epsilon, |M|\epsilon], 0) + I[-\delta, \delta],
$$

where, as before, $I[x, y]$ stands for a closed real interval with endpoints x and y, while the rest of the quantities are as in Lemma 5.3 and Prop. 5.2.

□

Now, part i) of Theorem 1 follows immediately from Proposition 5.2 and Lemmas 5.3 and 5.4.

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6 Compactness of renormalization

We will now outline the proof of compactness of the renormalization operator.

Proposition 6.1 *For every $s \in B_\delta(s_0) \subset \mathcal{A}(\rho)$ its renormalization $\mathcal{R}_{EKW}[s]$ is in $\mathcal{I}(\kappa\rho)$ with $\kappa = 1.0699996948242188..$*

Proof. Let $r' = \kappa r$ and $\rho' = \kappa\rho$. Verification that $|\lambda[s]|\rho' \le \rho$ is straightforward. The "difficult" part is to demonstrate that $|Z'_s|_{r'} \le \rho$. To show this, we will use the fact that $Z'_s = \lim_{n\to\infty} N_{s-s_0}[0]$,

To that end we first verify that $Z'_0 \in \mathcal{Z}(r')$. This amounts to checking on a computer that

$$|\upsilon_0| > |\upsilon|_{r'} \quad \text{and} \quad 1 - 4\frac{\upsilon_0\gamma_0}{\beta_0^2} > |F_1|_{r'} + |F_2|_{r'} \quad (\text{cf.} \quad (5.11)).$$

Below we will use the following shorthand notation: $t' \equiv |Z'_0|_{r'}$.

Set $z_0 = 0$, and define for all $n \ge 0$

$$z_{n+1} \equiv N_h[z_n], \quad \Delta z_n = z_{n+1} - z_n, \quad Z'_n = Z'_0 + M z_n$$

(we have suppressed the dependence on s in Z'_n for notational convenience), then

$$
\begin{aligned}
|\Delta z_0|_{r'} &= |C_h[Z'_0] - Z'_0|_{r'} \\
&\le \frac{2\delta\sqrt{2c}}{c - 4|a_0||\Theta_0|_{(r',t')} - 4|a_0|\delta} \quad (\text{cf.} \quad \text{equation} \quad (5.14)),
\end{aligned}
$$

and

$$
\begin{aligned}
|\Delta z_n|_{r'} &= |N_h[z_n] - N_h[z_{n-1}]|_{r'} \\
&= |\Delta z_{n-1} + C_h[Z'_{n-1} + M\Delta z_{n-1}] - C_h[Z'_{n-1}] - M\Delta z_{n-1}|_{r'} \\
&\le |\left[C_h[Z'_{n-1} + M\Delta z_{n-1}] - C_h[Z'_{n-1}] - DC_h[Z'_{n-1}]M\Delta z_{n-1}\right]|_{r'} + \\
&\quad |\left[DC_h[Z'_{n-1}] - DC_h[Z'_0]\right] M\Delta z_{n-1}|_{r'} \\
&\equiv T_1 + T_2.
\end{aligned}
$$

To evaluate T_2 we will require estimates similar to (5.15)–(5.18). In fact, estimates (5.15)–(5.18) can be recycled after one substitutes r' for r, ϵ' for ϵ and $s' \equiv t' + |M|\epsilon'$ for s (here ϵ' is an *a priori* bound on $|z_n|_{r'}$, $n \ge 0$, again, verifiable *a posteriori*). We will not repeat the details, but rather use the following symbolic notation

$$|T_2|_{r'} \le C_2 |\Delta z_{n-1}|_{r'},$$

keeping in mind that C_2 is computable on a machine.

To estimate T_1, we use a Cauchy bound for the following function:

$$
\begin{aligned}
F(\zeta) = \surd &c - 2a_0 \left[\Theta_0(\Pi_1, Z'_{n-1} + \zeta M\Delta z_{n-1}) + h(P_1, Z'_{n-1} + \zeta M\Delta z_{n-1}) + \right. \\
&\left. \overline{\Theta_0(\Pi_2, Z'_{n-1} + \zeta M\Delta z_{n-1}) + h(P_2, Z'_{n-1} + \zeta M\Delta z_{n-1})}\right],
\end{aligned}
$$

analytic on $\mathbb{D}_R$ with
$$R \le \frac{\rho - |Z'_{n-1}|_{r'}}{|M||\Delta z_{n-1}|_{r'}}.$$

This results in

$$
\begin{aligned}
|T_1|_{r'} &\le \frac{1}{R(R-1)} \frac{1}{2|a_0|} \sup_{|\zeta| \le R} |F(\zeta)|_{r'} \\
&= \frac{|M|^2 |\Delta z_{n-1}|^2_{r'}}{2|a_0|(\rho - s')(\rho - s' - |M||\Delta z_{n-1}|_{r'})} \sqrt{c + 4|a_0| \left[|\Theta_0|_{(r',s')} + \delta \right]} \\
&\equiv C_1 \frac{|\Delta z_{n-1}|^2_{r'}}{\rho - s' - |M||\Delta z_{n-1}|_{r'}}.
\end{aligned}
\tag{6.22}
$$

Finally, we have

$$
\begin{aligned}
|\Delta z_n|_{r'} &\le C_1 \frac{|\Delta z_{n-1}|^2_{r'}}{\rho - s' - |M||\Delta z_{n-1}|_{r'}} + C_2 |\Delta z_{n-1}|_{r'} \\
&= K_{n-1} |\Delta z_{n-1}|_{r'}, \\
K_{n-1} &\equiv C_1 \frac{|\Delta z_{n-1}|_{r'}}{\rho - s' - |M||\Delta z_{n-1}|_{r'}} + C_2
\end{aligned}
$$

(we check that $K_0 < 1$).

Clearly, K_n is a monotone increasing function of $|\Delta z_n|_{r'}$, therefore $K_n < K_{n-1}$. We can now verify that the sum

$$\Sigma \equiv \sum_{k=0}^{\infty} |\Delta z_k|_{r'}$$

is convergent:

$$
\begin{aligned}
\Sigma &\le \sum_{k=1}^{\infty} K_{k-1} |\Delta z_{k-1}|_{r'} + |\Delta z_0|_{r'} \\
&\le \sum_{k=2}^{\infty} K_{k-1} K_{k-2} |\Delta z_{k-2}|_{r'} + K_0 |\Delta z_0|_{r'} + |\Delta z_0|_{r'} \\
&\le \sum_{k=0}^{\infty} K_0^k |\Delta z_0|_{r'} \\
&\le \frac{|\Delta z_0|_{r'}}{1 - K_0}.
\end{aligned}
$$

Therefore, for all $n \ge 0$

$$|Z'_n - Z'_0|_{r'} \le |M| \frac{|\Delta z_0|_{r'}}{1 - K_0} \equiv \gamma'.$$

Finally, we verify that
$$t' + \gamma' < \rho.$$

$\square$

7   Acknowledgments

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