The Krein–Milman Theorem
A Project in Functional Analysis

Samuel Pettersson

November 29, 2016
Outline

1. An informal example
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2. Extreme points
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2. Extreme points
3. The Krein–Milman theorem
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3. The Krein–Milman theorem

4. An application
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1. An informal example

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3. The Krein–Milman theorem

4. An application
Convex sets and their “corners”

Observation
Some convex sets are the convex hulls of their “corners”.
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\[ \|x\|_1 \leq 1 \]
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\[ \|x\|_1 \leq 1 \quad \text{and} \quad \|x\|_2 \leq 1 \]
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Some convex sets are the convex hulls of their “corners”.

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\|x\|_1 \leq 1 \quad \|x\|_2 \leq 1 \quad \|x\|_\infty \leq 1
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Some convex sets are the convex hulls of their “corners”.

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Some convex sets are not the convex hulls of their “corners”.
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Observation
Some convex sets are **not** the convex hulls of their “corners”.

\[
x_1, x_2 \geq 0
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\[
\|x\|_\infty < 1
\]
Objectives

▶ Formalize the notion of a corner of a convex set (extreme point).
▶ Find a sufficient condition for a convex set to be the closed convex hull of its extreme points (Krein–Milman).
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- Find a sufficient condition for a convex set to be the closed convex hull of its extreme points (Krein–Milman).
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2. Extreme points
3. The Krein–Milman theorem
4. An application
Definition of an extreme point

Definition
An extreme point of a convex set $K \subseteq E$ in a vector space $E$ is a point $z \in K$ not in the interior of any line segment in $K$:

$$z \neq (1-t)x + ty, \quad \forall t \in (0,1), \forall x, y \in K, x \neq y$$

Remark
In a normed space,
▶ interior points are never extremal
▶ boundary points may be extremal
▶ boundary points (inside the set) of strictly convex sets are always extremal
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Examples of extreme points

Extreme points of the closed unit ball:
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Statement of Krein–Milman

Theorem (Krein–Milman)

A *compact* convex set $K \subseteq E$ in a normed space coincides with the closed convex hull of its extreme points:

$$K = \overline{\text{conv}}(\text{ext } K)$$
Statement of Krein–Milman

Theorem (Krein–Milman)

A compact convex set $K \subseteq E$ in a normed space coincides with the closed convex hull of its extreme points:

$$K = \overline{\text{conv}}(\text{ext } K)$$

Reminder

$\overline{\text{conv}} A := \overline{\text{conv}} A$

$= \text{the smallest closed and convex set containing } A$. 
Preparation for the proof: Extreme sets

Definition
Given a compact convex set $K \subseteq E$ in a normed space, an extreme set is a subset $M \subseteq K$ that is
- non-empty
- closed
- such that any line segment in $K$ whose interior intersects $M$ has endpoints in $M$:
  $$\exists t \in (0, 1): (1-t)x + ty \in M = \Rightarrow x, y \in M, \forall x, y \in K$$
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- such that any line segment in $K$ whose interior intersects $M$ has endpoints in $M$:

$$\exists t \in (0, 1): (1 - t)x + ty \in M \implies x, y \in M, \quad \forall x, y \in K$$
Lemma

For $A \subseteq K$ an extreme set and $f \in E^*$,

$$B := \{ x \in A : \langle f, x \rangle = \max_{y \in A} \langle f, y \rangle \} = \{ \text{maxima of } f \text{ on } A \}$$

is an extreme subset of $K$.
Lemma

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Proposition
Every extreme set $A \subseteq K$ contains an extreme point of $K$. 
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is an extreme subset of $K$.

Proposition
Every extreme set $A \subseteq K$ contains an extreme point of $K$.

Proof.
Use Zorn’s lemma and the above lemma (details omitted).
Theorem (Krein–Milman)

A compact convex set $K \subseteq E$ in a normed space coincides with the closed convex hull of its extreme points:

$$K = \overline{\text{conv}(\text{ext } K)}$$
Proof of Krein–Milman

For \( \text{conv}(\text{ext } K) \subseteq K \),
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Proof of Krein–Milman

For \( \text{conv}(\text{ext } K) \subseteq K \),

\( K \) compact, convex, and \( K \supseteq \text{ext } K \)

\( \implies K \) closed, convex, and \( K \supseteq \text{ext } K \)
Proof of Krein–Milman

For $\overline{\text{conv}}(\text{ext } K) \subseteq K$,

\[ K \text{ compact, convex, and } K \supseteq \text{ext } K \]
\[ \implies K \text{ closed, convex, and } K \supseteq \text{ext } K \]
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For $K \subseteq \text{conv}(\text{ext } K)$,
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$$K = \emptyset \implies K \subseteq \overline{\text{conv}}(\text{ext } K)$$

Otherwise, argue by contradiction:

$$\exists x \in K \setminus \overline{\text{conv}}(\text{ext } K) \iff \exists f \in E^\star: f(\overline{\text{conv}}(\text{ext } K)) < f(x) \ (\text{Hahn–Banach}) \Rightarrow \exists f \in E^\star: f(\text{ext } K) < f(x) \Rightarrow \exists B \subseteq K \text{ extreme set without extreme points (Lemma)} \Rightarrow \text{Contradiction! (Proposition)}$$

Hence, $K \subseteq \overline{\text{conv}}(\text{ext } K)$.
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Hence,
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$$\implies \exists B \subseteq K \text{ extreme set without extreme points} \quad \text{(Lemma)}$$
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\[\implies \text{Contradiction!} \] \hspace{1cm} \text{(Proposition)}

Hence, $K \subseteq \overline{\text{conv}(\text{ext } K)}$ \hfill $\square$
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Not dual spaces

Example:

\[ c_0 \subseteq \ell_\infty \text{ and } L_1(\mathbb{R}) \text{ are not dual spaces.} \]

**Proposition**

The closed unit ball \( B_E^\star \) has an extreme point.

**Proof.**

\( B_E^\star \) is weakly \( \star \) compact (Banach–Alaoglu–Bourbaki) \( \Rightarrow \)

\( B_E^\star = \text{conv}(\text{ext } B_E^\star) \) (generalized Krein–Milman) \( \Rightarrow \)

\( B_E^\star \) has an extreme point.
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The closed unit ball $B_{E^*}$ has an extreme point.
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*The closed unit ball $B_{E^*}$ has an extreme point.*

Proof.
$B_{E^*}$ is weakly* compact \hspace{1cm} \text{(Banach–Alaoglu–Bourbaki)}

$\implies B_{E^*} = \overline{\text{conv}}(\text{ext } B_{E^*})$ \hspace{1cm} \text{(generalized Krein–Milman)}

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