Positive Linear Functionals

Problem 5 in Brezis

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Outline

Introduction
  Definitions
  Goal

Proof
  Derive (ii) by (i)
  Derive (i) by (ii) when $P$ is closed and $E$ is complete.

Examples
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Examples
Convex cone

**Definition**
Let $E$ be an n.v.s. $P$ is a convex cone with vertex at 0 if $P$ is closed under linear combinations with positive coefficients, i.e., $\lambda x + \mu y \in P$, $\forall x, y \in P$, $\forall \lambda, \mu > 0$. 

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Examples
Let $F$ be an n.v.s. and let $P$ be a convex cone with vertex at 0. Set $F = P - P$, so that $F$ is a linear subspace.
Let $F$ be an n.v.s. and let $P$ be a convex cone with vertex at 0. Set $F = P - P$, so that $F$ is a linear subspace. Consider the following two properties:

(i) Every linear functional $f$ on $E$ such that $f(x) \geq 0 \ \forall x \in P$, is continuous on $E$.

(ii) $F$ is a closed subspace of finite codimension.

Our goal is to prove:

$\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (i) when $E$ is a Banach space and $P$ is closed.
Let $F$ be an n.v.s. and let $P$ be a convex cone with vertex at 0. Set $F = P - P$, so that $F$ is a linear subspace. Consider the following two properties:

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- $(i) \implies (ii)$
- $(ii) \implies (i)$ when $E$ is a Banach space and $P$ is closed.
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Examples
Previous conclusion

Ex. 1.5
Let $E$ be an infinite-dimensional n.v.s. There exists an algebraic basis of $E$. What’s more, if $\{e_\alpha\}_{\alpha \in \Gamma}$ is a set of linear independent vector in $E$, we can expand it to a set of algebraic basis of $E$. [Proved by Zorn’s Lemma.]
And we can always construct a linear functional on $E$ which is not continuous.
Step 1: $F$ is closed.

Through this part we assume (i), i.e., for every linear functional $f$ on $E$, if $f$ is non-negative on $P$, then $f$ is continuous on $E$.

**Proof.**

We assume, by contradiction, that there exists $\{x_n\} \subset F$, such that $\lim_{n \to \infty} x_n = x_0 \notin F$.

Then construct a functional $f$ on $E$ by following steps:

- For each $w \in E$, take "the component of $w$ in the direction of $x_0" : kx_0$
- Consider the functional on $E$ defined by $f(w) = k$. 

\[
\boxed{}
\]
Step 2: Every linear subspace of $E$ of which the intersection with $F$ contains only zero is of finite dimension.

Proof.

We assume, by contradiction, that there exists a linear infinite-dimensional subspace $M \subset E$, such that $M \cap F = \{0\}$. We may take a $M' \subset M$ of countable dimension with basis $\{f_n\}_{n \in \mathbb{N}}$. Then construct a functional $\phi$ on $E$ by following steps:

- For each $w$ in $E$, take "the component of $w$ in $M'$":
  \[ \sum_{n \in \Gamma} a_n f_n. \]
- Consider the functional $\phi$ on $E$ defined by:
  \[ \phi(w) = \sum_{n \in \Gamma} n a_n y_n \]
Step 3: The quotient space $F/E$ is of finite dimension.

Proof.
Take a set of basic of $E$: $\{e_\alpha\}_{\alpha \in \Gamma}$. Then we could take the set of representation element of each of those equivalent class:

$$\{\hat{e}_\alpha\}_{\alpha \in \Gamma} \subset E$$

Let $M$ be the linear subspace spanned by $\{\hat{e}_\alpha\}_{\alpha \in \Gamma} \subset E$. Then $M$ is of finite dimension by previous conclusion. Therefore, $F/E$ is of finite dimension, namely, $F$ is of finite codimension. $$\square$$
Outline

Proof

Derive (ii) by (i)

Derive (i) by (ii) when $P$ is closed and $E$ is complete.
First step: when $P - P = E$

Through this part we assume (ii), i.e., $F$ is a closed subspace of finite codimension. We also assume that $E$ is a Banach space and $P$ is closed.

The thought is to prove:

(a) There exists a constant $C > 0$ such that every $x \in E$ has a decomposition $x = z - y$ with $y, z \in P$, $\|y\| \leq C\|x\|$ and $\|z\| \leq C\|x\|$.

(b) Argue by contradiction.
Step 1: (a)

Claim that there exists a constant $C > 0$ such that every $x \in E$ has a decomposition $x = z - y$ with $y, z \in P$, $\|y\| \leq C\|x\|$ and $\|z\| \leq C\|x\|$.

Consider the set:

$$K = \{ x = y - z \text{ with } y, z \in P, \|y\| \leq C\|x\| \text{ and } \|z\| \leq C\|x\| \}$$

Then it is suffice to prove that there exists a constant $c$ such that $B(0, c) \subset K$.

1. Find a $c$ and $y_0 \in E$ such that $B(y_0, 4c) \subset \bar{K}$.
2. Show that $B(0, 2c) \subset \bar{K}$.
3. Show that $B(0, c) \subset K$. 
Step 1: (b)

Consider a sequence \((x_n)\) in \(E\) such that \(\|x_n\| \leq \frac{1}{2^n}\) and \(f(x_n) \geq 1\). Then we have:

\[
x_n = y_n - z_n, \quad \|y_n\| \leq C \|x_n\| \quad \text{and} \quad \|z_n\| \leq C \|x_n\|.
\]

Set \(u_n = \sum_{i=1}^{n} y_i\) and \(u = \sum_{i=1}^{\infty} y_i\).
Therefore,

\[
P \ni u - u_n = \sum_{i=n+1}^{\infty} y_i \implies f(u) \geq f(u_n), \quad \forall \, n \in \mathbb{N}
\]

\[
1 \leq f(x_n) \leq f(y_n) \implies f(u_n) \geq n, \quad \forall \, n \in \mathbb{N}.
\]

Hence we arrive at a contradiction.
Step 2: general case

By example 2 in Section 2.4, $F$ admits a complement $M$ of finite dimension.
Then we have:

- $f$ is continuous on $F$. (Apply the previous conclusion to $F$)
- $f$ is continuous on $M$. ($M$ is a finite-dimensional linear subspace)
- For each $x$ in $E$, consider its decomposition.
Examples

Check whether (i) or (ii) holds in the following examples.

(a) \( E = C([0, 1]) \) with its usual norm and
    \[ P = \{ u \in E; u(t) \geq 0, \forall t \in [0, 1] \} , \]

(b) \( E = C([0, 1]) \) with its usual norm and
    \[ P = \{ u \in E; u(t) \geq 0, \forall t \in [0, 1] \text{ and } u(0) = u(1) = 0 \} , \]

(c) \( E = \{ u \in C([0, 1]); u(0) = u(1) = 0 \} \) with its usual norm and
    \[ P = \{ u \in E; u(t) \geq 0, \forall t \in [0, 1] \} , \]

(d) \( E = C([0, 1]) \) with the norm: \( \| f \| = \int_0^1 |f(t)| dt, \forall f \in E \)
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(c) \( E = \{ u \in C([0, 1]); u(0) = u(1) = 0 \} \) with its usual norm and
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(d) \( E = C([0, 1]) \) with the norm:
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(d) shows that we can not derive (i) by (ii) when \( E \) is not complete.