# Ph.D. course in geometry & topology

## §I. Introductory notions

One goal of topology is to understand the following categories

| | Ob | Mor | Iso$^s$ |
|---|---|---|---|
| Top | sets $X$ equipped with a topology $\Omega_X \subseteq \mathcal{P}(X)$ | $f: X \to Y$ cont., i.e $U \in \Omega_Y \Rightarrow f^{-1}(U) \in \Omega_X$ <br> write: $f \in C(X,Y)$ | homeomorphisms <br> $f \in C(X,Y)$ s.t. $f^{-1} \in C(Y,X)$ <br> (cont. bij. w. cont. inv.) |
| hTop | —‖— | $[X,Y] := C(X,Y)/\sim$ <br> $\sim$ = homotopy <br> (later today) | homotopy equiv's <br> $f \in C(X,Y)$ s.t. $\exists g \in C(Y,X)$ & $f \circ g \sim \mathrm{id}_Y$, $g \circ f \sim \mathrm{id}_X$ |
| $\mathrm{Man}^p \subseteq \mathrm{Top}$ | $X$ $C^p$-manifold <br> ($p = 0, 1, \dots, \infty, \omega$) <br> analytic | $C^p(X,Y) \subseteq C(X,Y)$ <br> $C^p$-smooth maps | $p = \infty$: <br> diffeomorphisms <br> i.e. smooth bij. w. smooth inv. |

We will only consider topological spaces whose topology is induced by an auxiliary metric $d$

i.e. $$\Omega = \left\{ \begin{array}{c} \text{arbitrary unions of open balls} \\ B_r(pt) = \{ x \in X \mid d(x, pt) < r \} \end{array} \right\} \subseteq \mathcal{P}(X)$$

where

**Def** a metric satisfies

$$d : X \times X \longrightarrow [0, +\infty)$$

(M1) $d(x,y) = 0 \Leftrightarrow x = y$ (non-degen.)

(M2) $d(x,y) = d(y,x)$ (symmetry)

(M3) $d(x,z) \leqslant d(x,y) + d(y,z)$ (triangle ineq.)

## The homotopy relation

An equivalence relation on the set $C(X,Y)$.

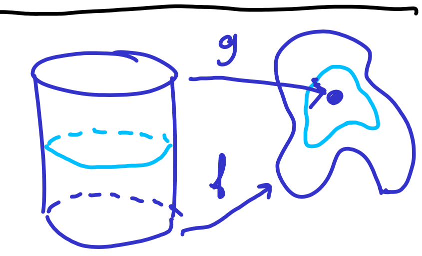
**Def** $f, g \in C(X,Y)$ are homotopic $(f \sim g)$ if

prod. top.

$\exists$ $F \in C(X \times [0,1], Y)$ s.t.

euclidean topology

$$F|_{X \times \{0\}} = f \quad \& \quad F|_{X \times \{1\}} = g$$




the simple closed curve $f$ is htpic to a cst map.

**Def** A space $X$ is <u>contractible</u> if it is isomorphic to $\{pt\}$ in $\underline{hTop}$.

**Exercise 0.** $\mathbb{R}^n$ is contractible

euclidean topology

**Def.** An $n$-dimensional <u>$C^0$ (topological) manifold</u> is a $2^{nd}$ countable metric space $X$ which is locally homeomorphic to $\mathbb{R}^n$, i.e

countable fam. of balls generate $\Omega$

for all $pt \in X$ $\exists$ $\underset{\underset{pt}{\in}}{U}^{open} \subseteq X$ s.t. $U \underset{homeo}{\cong} \mathbb{R}^n$

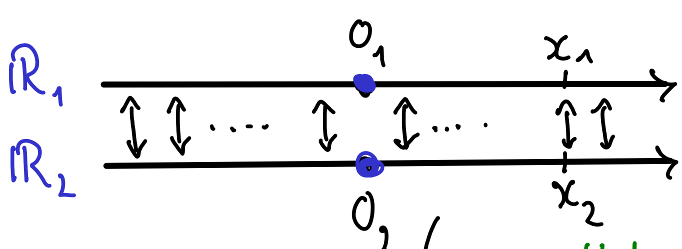
A manifold is called <u>closed</u> if it is connected & compact

**Ex.** 1.) Any open subset of $\mathbb{R}^n$ (obs: $B^n \cong \mathbb{R}^n$) is an $n$-mfd.

2.) $S^n := \{\bar{x} \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}$ is a closed $n$-dim. mfd $(n = 0,1,2,\dots)$

$\boxed{\textbf{Def } N := (0,\dots,0,1) \text{ (north-pole)}}$

**Non-ex.** A one-dim non-Hausdorff "manifold":




form the topological quotient $x_1 \sim x_2$ unless $x = 0$

(all nbhds. of $0_1$ & $0_2$ intersect: $0_1$ $0_2$ $0_1$ $0_2$)

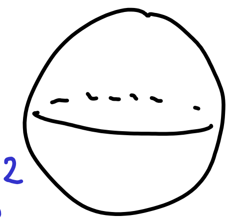
**Exercise 1.)*** Show that all closed 1-dim mfds are homeom. to $S^1$.

* technical

## Möbius & Jordan (1880s) classified the closed surfaces (2-dim mfds)

they are classified by their fundamental groups $\pi_1$

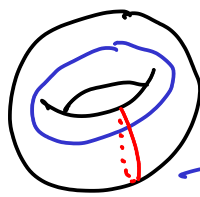
**oriented**



$S^2$

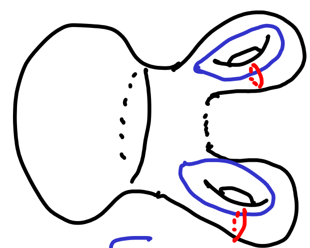
sphere

$\pi_1(S^2, pt) = \{0\}$



torus

$T^2 \cong S^1 \times S^1$

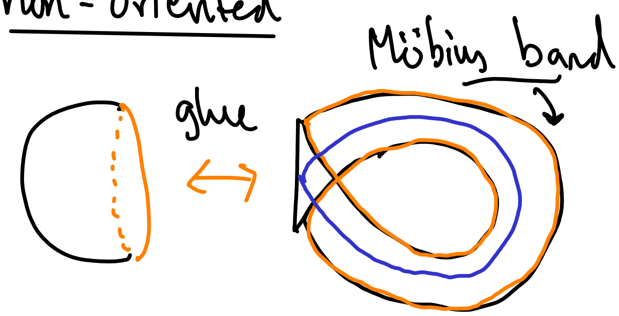


$\#g$ "handles"

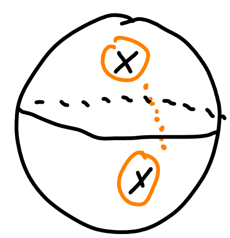
$\Sigma_g$

genus $g$ surface

**non-oriented**



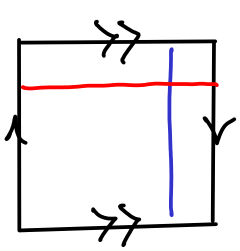

$\mathbb{R}P^2$ real proj. plane



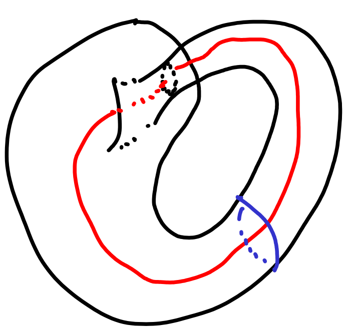
$\# k$ of Möbius bands

$N_k$

**Exercise 2.)** a) Show that $N_2$ is homeomorphic to

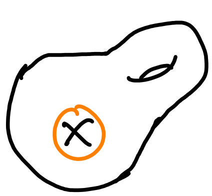


(glue edges)

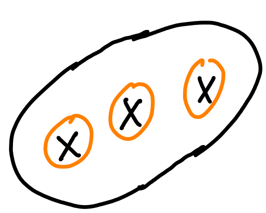


Klein bottle

b) Show that



$\cong$



**The Poincaré conjecture** (Perelman '04) If $X$ is a simply connected closed $3$-mfd, then $X \cong S^3$
i.e. $\pi(X, pt) = \{0\}$
(homeomorphic, even diffeomorphic).

- We will prove the 2-dim. case later
- The $n$-dim case, $n \geq 5$, was proven by Smale in the 60's, but we need to assume more ($\pi_k(X) = 0 \quad k < n$)

# §II. Homotopy groups

The homotopy classes of maps from spheres contains a lot of information.

To obtain an algebraic structure, we need "pt constraints".

$\underline{Top_\star}$, $\underline{hTop}_\star$ categories of $\underline{\text{pointed}}$ spaces, i.e.

$\underline{\text{Ob}}$: $(X, pt_X)$, $pt_X \in X$ a choice of "basepoint"

$\underline{\text{Mor}}$: $f: X \to Y$ s.t. $f(pt_X) = pt_Y$, write $f \in C((X, pt_X), (Y, pt_Y))$

$\sim_\star$ : homotopy through basepoint preserving maps

$$[(X, pt_X), (Y, pt_Y)]_\star := C((X, pt_X), (Y, pt_Y))/\sim_\star$$

$\underline{\underline{\text{Def.}}}$ The $\underline{n\text{:th homotopy group}}$, $k = 0, 1, 2, \dots$ is

$N = (0, \dots, 0, 1)$ north pole

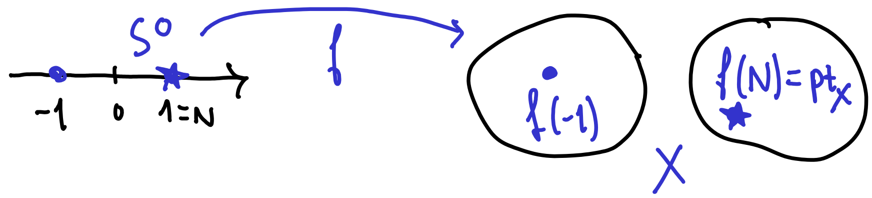
$$\boxed{\pi_k(X, pt_X) := [(S^k, N), (X, pt_X)]_\star}$$

(i.e. $\pi_k$ is the hom-functor $\underline{hTop_\star} \longrightarrow \underline{Set}$

$(X, pt_X) \longmapsto hom((S^k, N), (X, pt_X))$ )

## 1.) $\pi_0(X, pt_X)$: the set of path components of $X$

For manifolds, same as the set of <u>conn. components</u>




This is not a group in general, <u>but</u> it is e.g.

when $X = G$ is a <u>topological group</u>, i.e.

(G1) $\exists\ \mu : G \times G \xrightarrow{\text{cont.}} G$ (mult.)

$\qquad (g_1, g_2) \longmapsto g_1 \cdot g_2$

(G2) $\exists\ e \in G$ s.t. $e \cdot g = g \cdot e = g \quad \forall g \in G$ (unit)

(G3) $\exists\ G \xrightarrow{\text{cont.}} G$ s.t. $g \cdot g^{-1} = g^{-1} \cdot g = e$ (inverse)

$\qquad g \longmapsto g^{-1}$

(G4) $\forall\ g_1, g_2, g_3 \in G$ : $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$ (assoc.)

**Prop 1.** $\pi_0(G, e)$ is a group w. unit $\left[\begin{Bmatrix} -1 \mapsto e \\ 1 \mapsto e \end{Bmatrix}\right]$ & mult. induced by

$$\left[\begin{Bmatrix} -1 \mapsto g_1 \\ 1 \mapsto e \end{Bmatrix}\right] \cdot \left[\begin{Bmatrix} -1 \mapsto g_2 \\ 1 \mapsto e \end{Bmatrix}\right] := \left[\begin{Bmatrix} -1 \mapsto g_1 \cdot g_2 \\ 1 \mapsto e \end{Bmatrix}\right]$$

<u>Example</u> $Gl_n(\mathbb{R}) := \{A \in Mat_{n,n}(\mathbb{R}) \mid \det A \neq 0\}$ is a top. group w. $\mu$ = matrix mult. (Since $Gl_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ is an open subset, it is in fact an $n^2$-dim. <u>manifold</u>.)

<u>Exercise 3.)</u> Show that

$$\pi_0(Gl_n(\mathbb{R}), I) \longrightarrow \{\pm 1\}$$
$$\left\{\begin{matrix} -1 \mapsto A \\ 1 \mapsto I \end{matrix}\right\} \longmapsto \frac{\det A}{|\det A|}$$

is an iso of gp^s.

<u>Hint:</u> argue by induction on $n$, normalise the matrix $A$ to the form $\begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \boxed{A} \\ 0 & \end{bmatrix}$ by a cont. realisation of Gauss' alg.