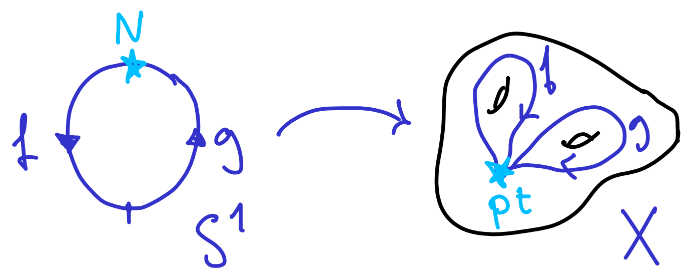
2.) $\pi_1(X, pt)$: the fundamental group

This is always a group, with

<u>unit</u> = $[\text{cst. map } S^1 \to \{pt\} \subseteq X]$

<u>mult</u>: concatenation $[f]\cdot[g] \overset{\text{def}}{=} [f * g]$



<u>inverse</u>: backwards param., alt. precomp. with reflection

- group axioms only <u>up to homotopy</u>
- in general this group is not abelian

<u>Def.</u> X is <u>simply connected</u> if $\pi_0(X, pt) = \pi_1(X, pt) = \{0\}$

What about the dependence on $pt \in X$?

We will use a more general (and abstract) concept

the <u>fundamental groupoid</u> $\Pi_1(X)$ is a category with

<u>Obj.</u> points $pt \in X$

<u>Mor.</u> $\mathrm{Hom}(pt_0, pt_1) = \left\{ \begin{array}{l} \text{cont. paths } \gamma : [0,1] \longrightarrow X \\ \gamma(0) = pt_0,\ \gamma(1) = pt_1 \end{array} \right\} \Big/ \begin{array}{l}\text{htpy through} \\ \text{such paths}\end{array}$

composition: concatenation (up to htpy) $\quad f(t) \star g(t) = f(2t)\ g(2t-1)$

<u>Basic facts:</u>

(by def. of π_0)

- $\exists$ morphism $pt_0 \longrightarrow pt_1 \iff [pt_0] = [pt_1] \in \pi_0(X)$
- $\mathrm{Aut}(pt) = \pi_1(X, pt)$
- All morphisms are <u>isomorphisms</u> ("groupoid")

In particular, we deduce from "algebra"/"cat. thy.":

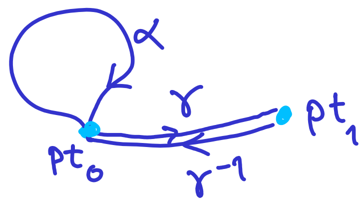
<u>**Prop. 3**</u> If $[pt_0] = [pt_1] \in \pi_0(X)$ then there exists a isomorphism of groups $\phi : \pi_1(X, pt_0) \xrightarrow{\cong} \pi_1(X, pt_1)$ canonically defined up to an inner automorphism (i.e. conjugation by an element in the group). $\quad g \mapsto hgh^{-1}$

(π_1 abelian $\Rightarrow$ truly canonical!)

Proof.

Choose $[\gamma]: pt_0 \to pt_1 \rightsquigarrow \phi([\alpha]) := [\gamma]\circ[\alpha]\circ[\gamma^{-1}]$

comp. in Hom_{Π_1}



$\square$

When X is a conn. manifold, $pt \in X$, its <u>universal cover</u> is

$$\tilde{X}_{pt} := \{\gamma \in \mathrm{Hom}(pt, x) \mid x \in X\} \xrightarrow[\text{surj.}]{\pi} X$$

$$\gamma \longmapsto \gamma(1)$$

endowed with the <u>initial topology</u> w.r.t. X & π

<u>Later:</u> $\tilde{X}_{pt}$ is simply connected, $\pi^{-1}(x) \cong \pi_1(X, x)$

<u>Recall</u> $f: A \longrightarrow B$ a morphism between sets

- <u>initial topology on A</u> w.r.t a topology Ω_B on B

$V \subseteq B$ open $:\Longleftrightarrow$ $V = f^{-1}(U)$, $U \subseteq B$ open

(the "smallest" topology on A for which f is cont.)

- <u>final topology on B</u> w.r.t. a topology Ω_A on A

$U \subseteq B$ open $:\Leftrightarrow$ $f^{-1}(U) \subseteq A$ is open ("largest" top on B...)

When f is surj. this is called the <u>quotient topology</u> on $B = A/_{x_1 \sim x_2}$ $x_1 \sim x_2 \Leftrightarrow f(x_1) = f(x_2)$

"glueing" of spaces

(in this case: $\Omega_B = \Omega^{final}_{(A, \Omega^{init}_B)}$)

<u>Exercise 4.)</u>* Assume that X is a connected topological manifold with $\pi_1(X, pt)$ countable.

Show that $\widetilde{X}_{pt}$ is a topological manifold.

<u>Def</u> The first homology group of a conn. mfd. X is the abelian group

$$H_1(X, \mathbb{Z}) := \pi_1(X, pt) / Comm(X)$$

(no dep. on pt by Prp. 3!)

normal subgp. gen. by $ghg^{-1}h^{-1}$

<u>Exercise 5.)</u> Show that $\pi_1(G, pt_1) \cong \pi_1(G, pt_2)$ for any $pt_1, pt_2 \in G$ when G is a (possibly disconnected) mfd. which admits the structure of a topological group.

Prop 4. If G is a topological group, then $\pi_1(G,e)$ is abelian.

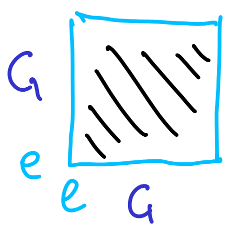
(In view of Prp. 3 & Exc. 5: suff. for $\pi_1(G,pt)$)

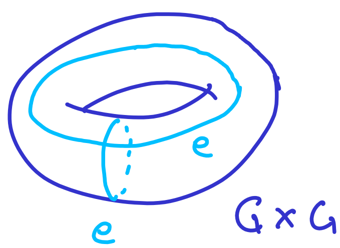
Ex $G = SO(2) = U(1) = S^1 \subseteq \mathbb{C}^* = Gl_1(\mathbb{C})$

$G = U(1)^n = (S^1)^n \subseteq (\mathbb{C}^*)^n \subseteq Gl_n(\mathbb{C})$

Proof Let $\alpha, \beta : S^1 \to G$ be two paths,

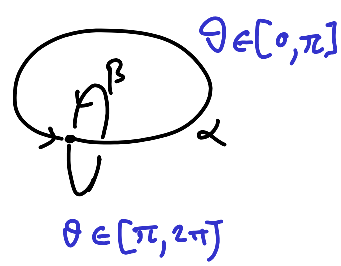
$\alpha(N) = \beta(N) = e$.





cont!

$G\times G \xrightarrow{\mu} G$

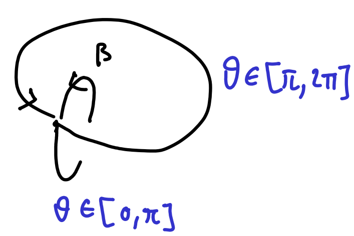


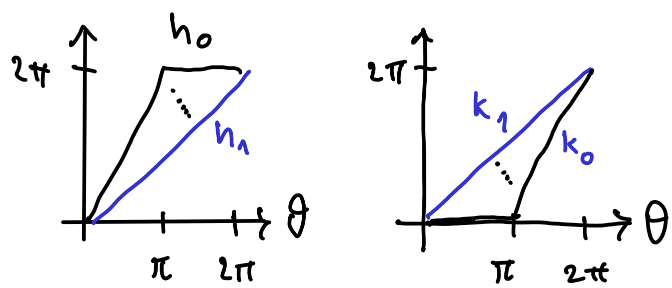
$$\alpha\star\beta : S^1 \xrightarrow{(\alpha\circ h_0,\,\beta\circ k_0)} \underset{\alpha}{G\times\{e\}} \cup \underset{\beta}{\{e\}\times G} \longrightarrow G$$

htpic

$$S^1 \longrightarrow G$$
$$\theta \longmapsto \mu(\alpha(\theta), \beta(\theta)) = \mu(\alpha\circ k_1, \beta\circ h_1)$$

$$\beta\star\alpha : S^1 \xrightarrow{(\alpha\circ k_0,\,\beta\circ h_0)} \underset{\beta}{\{e\}\times G} \cup \underset{\alpha}{G\times\{e\}} \longrightarrow G$$





homotopic "reparametrisations" of the domain

$\square$

3. Higher homotopy groups

$\pi_k(X, pt) = [(S^k, N), (X, pt)]$ has an <u>abelian</u> gp. structure

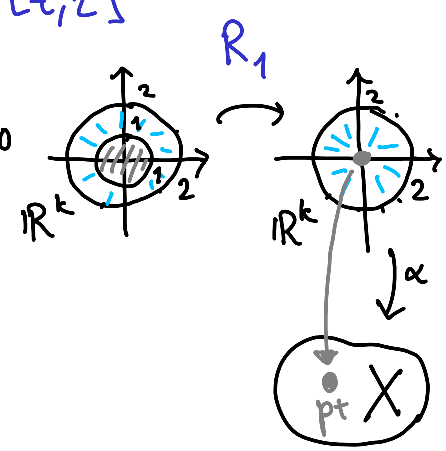
<u>unit</u>: $0 = [\text{cst. map } S^k \to \{pt\} \subseteq X]$

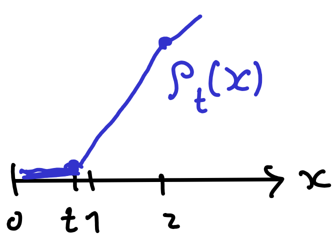
<u>mult</u>: represent $[\alpha], [\beta] \in \pi_k(X, pt)$ by maps α, β which are cst$^{\text{ly}}$ $= pt$ in a <u>nbhd</u> of N

<u>Key technique</u>: Recall $S^k \setminus \{P\} \cong \mathbb{R}^k$ for any $P \in S^k$

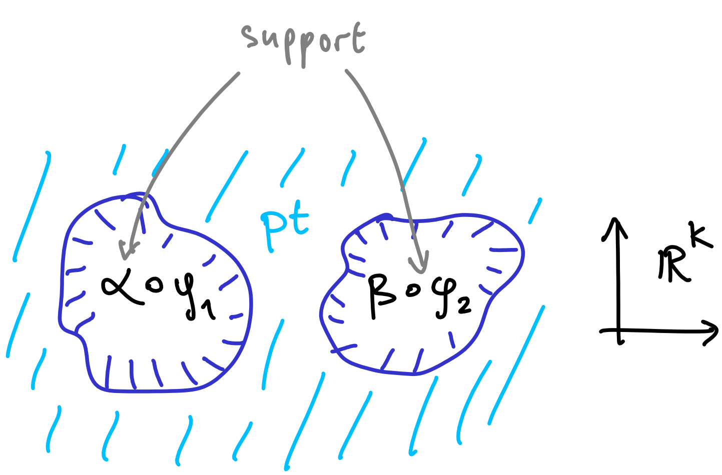
Construct $\quad \rho_t(x) = \begin{cases} x, & x \geq 2 \\ 0, & x \leq t \\ 2(x-t)/2-t, & x \in [t, 2] \end{cases} \quad t \in [0,1]$

$\mathbb{R}^k \ni \bar{x} \mapsto R_t(\bar{x}) = \dfrac{\rho_t(\|\bar{x}\|)}{\|\bar{x}\|} \cdot \bar{x}$ htpy. from $id_{\mathbb{R}^k}$ to



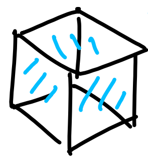


$[\alpha] \cdot [\beta] \overset{\text{def.}}{=}$



after precomposing w. <u>suitable</u> linear translations $\varphi_1, \varphi_2 : \mathbb{R}^k \to \mathbb{R}^k$ of the domain to disjoin the supports

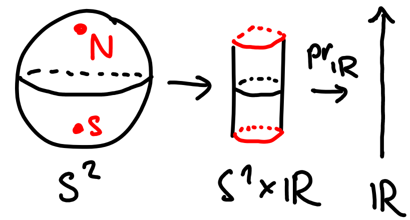
Alt. def. $\pi_k(X, pt) =$



(concatenation = "stacking" of cubes)

inverse : $[\alpha]^{-1} = [\alpha \circ \varphi]$ where $\varphi : \mathbb{R}^k \to \mathbb{R}^k$ reflection (or reflect. $S^k \to S^k$)

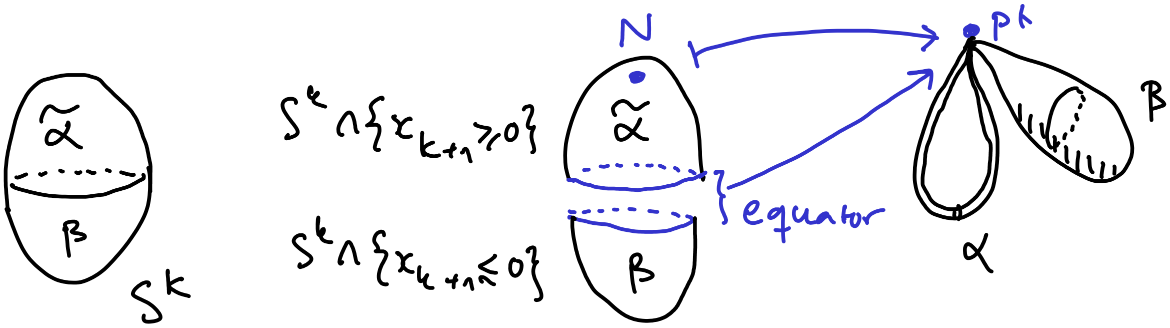
Recall: $S^k \setminus \underbrace{\{|x_{k+1}| = 1\}}_{\{N, S\}} \cong S^{k-1} \times$



$\alpha : \mathbb{R} \to X \rightsquigarrow \tilde{\alpha} := \alpha \circ pr_{\mathbb{R}} : S^k \setminus \{|x_{k+1}| = 1\} \longrightarrow X$

canonical proj.

Given $[\alpha] \in \pi_1(X, pt)$ & $[\beta] \in \pi_k(X, pt)$ we can form an element in $\pi_k(X, pt)$.



This induces a $\mathbb{Z}$-linear group action (exc. 6.c. below)

$$\pi_1(X, pt) \circlearrowright \pi_k(X, pt), \quad k \geq 2.$$

($\Leftrightarrow$ a $\mathbb{Z}[\pi_1(X, pt)]$-module structure)

Exercise 6.) Let X be connected. Show that

a) forgetting the basepoint
$$\pi_k(X, pt) = [S^k, X]_* \longrightarrow [S^k, X]$$
is a surjective map of sets.

b) When $k=1$ this map is injective on the conjugacy classes of $\pi_1(X, pt)$.

c) Show that π_1 acts linearly on π_k.

When $k \geq 2$ the above map is injective on
$$\pi_k(X, pt) / \pi_1(X, pt)$$