4. Computations

Generalised Poincaré conj: X closed n-dim. mfd. $\pi_k(X)=0$, $k<n$,
$\Rightarrow X \cong S^n$ (homeo) (Perelman $n=3$, Smale $n\geq 5$, Freedman $n=4$)

Prop. 6. $\pi_k(S^n, N) = \begin{cases} \mathbb{Z}_2, & k=n=0 \\ 0, & k=0,\ k<n, \\ \mathbb{Z}, & 0<k=n \end{cases}$

Proof Case $k=0$: (Obs: $S^0 \cong \mathbb{Z}_2$ is a group)

$n=0$: by hand: $\bar{0} = \left\{ \begin{matrix} N \mapsto N \\ S \mapsto N \end{matrix} \right\}$ & $\bar{1} = \left\{ \begin{matrix} N \mapsto N \\ S \mapsto S \end{matrix} \right\}$

$n>0$: for any $x \in S^n$, we can construct a path from N to pt. In $S^n \setminus \{N\} = \mathbb{R}^n$ consider $\frac{1}{1-t} \cdot \bar{x}_{pt} \in \mathbb{R}^n$, $t \in [0,1)$
Compactify $\rightsquigarrow [0,1] \to S^n$

In order to compute $\pi_k(S^n, N)$ for $1 \leq k \leq n$ we will triangulations & piecewise linear approximations.

We begin with a simple but useful lemma.

Lem. 7 Let X be a metric space w. a choice of metric. Two maps $f, g \in C(X, S^n)$ which are sufficiently C^0-close ($\sup_{x\in X} d(f(x), g(x))$) are homotopic, where the homotopy moreover can be taken to be fixed in the (closed) subset $\{f(x) = g(x)\} \subseteq X$.

by Hausdorff prop.

<u>Proof.</u> Consider the "convex" interpolation

$$F(x,t) := \frac{(1-t)f(x)+t\,g(x)}{\|(1-t)f(x)+t\,g(x)\|}$$

$\square$

By the above lemma, we are able to apply different smoothing techniques to the case when $X = \mathbb{R}^k$ or $\mathbb{R}^k \times \mathbb{R}$ in order to replace an arbitrary

- map $C(S^k, S^n)$, or
- homotopy $C(S^k \times [0,1], X)$

by one which is better behaved (e.g. smooth/piecewise linear.)

Continuity is not sufficiently well-behaved, due to:

$\exists$ continuous "space-filling curve"

$$[0,1] \xrightarrow{\text{surj.}} [0,1]\times[0,1]$$

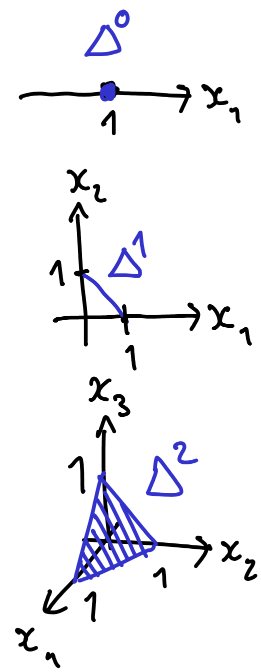
See [Armstrong; Basic Topology] or [Munkres; Topology].

Obviously no piecewise lin. such curve (measure theory!) (Sard: no such curve of class C^1 exists)

Simplicial complexes & Triangulations

We will do a piecewise lin. approximation by hand.
For that reason, we now introduce simplicial complexes.
The n-dim^l simplex is the topological space

$$\Delta^n := \{ x_1 + \dots + x_{n+1} = 1 \mid x_i \geqslant 0 \} \subseteq \mathbb{R}^{n+1}$$

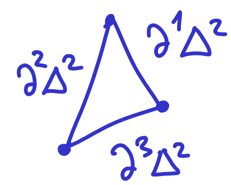


There are $\binom{n}{k}$ "linear" subsimplices of dim. $n-k$:

$$\Delta^{n-k} \cong \underbrace{\Delta^n \cap \{ x_{i_1} = \dots = x_{i_k} = 0 \}}_{\text{lin. sub } n-k\text{-simplex det. by } i_1 < \dots < i_k}$$
(*)

$$\partial^i \Delta^n := \Delta^n \cap \{ x_i = 0 \}, \quad i = 1, \dots, n+1,$$

the subsimplices of dim $= n-1$



$\partial \Delta^n := \bigcup \partial^i \Delta^n$ the boundary of $\Delta^n \cong S^{n-1}$

(*) Obs: $\{$ordered choice of a nr. $n-k+1$ of $\overbrace{0\text{-subsimplices}}^{\text{vertices}}\} =$

$\{$Linear inclusion $\Delta^{n-k} \subseteq \Delta^n$ as a subsimplex$\}$

the lin. inclusion $\phi : \mathbb{R}^{n-k+1} \to \mathbb{R}^{n+1}$ corr. to the ordered sequence

$$\underbrace{\phi(1,0,\dots,0),\ \phi(0,1,0,\dots,0),\ \dots,\ \phi(0,0,\dots,1)}_{n-k+1} \in \Delta^n$$ of vertices.

A <u>simplicial complex</u> is the combinatorial description of a space obtained by gluing simplices together:

V = set of vertices (here: <u>finite</u>)

S = collection of finite non-empty subsets of V

s.t. • $\forall v \in V: \{v\} \in S$

• $F \in S \Rightarrow \mathcal{P}(F) \setminus \{\emptyset\} \subseteq S$

(<u>think</u>: $F \in S \Rightarrow$ vertices in F span an $(|F|-1)$-dim simplex)

Out of V & S we can construct a (metric) top. space X by:

0-dim skeleton

Let $X^0 = \coprod_{|F|=1} \Delta^0 \times \{F\} = \coprod_{v \in V} \Delta^0 \times \{\{v\}\}$ w. discrete topology

Argue by induction

⋮

Assume X^{k-1} satisfies $|G| \leq k \Rightarrow \Delta^{|G|-1} \overset{\iota_G}{\hookrightarrow} X^{k-1}$ "lin." inclusion

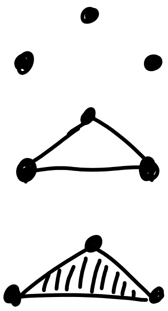
$$X^k = X^{k-1} \sqcup \coprod_{|F|=n+1} \Delta^k \times \{F\} \Big/ \sim$$

k-skeleton

where $\sim$ induced by inclusions $\partial\Delta^k \times \{F\} \overset{\iota}{\hookrightarrow} X^{k-1}$

s.t. • lin. on each $\partial^i \Delta^k$

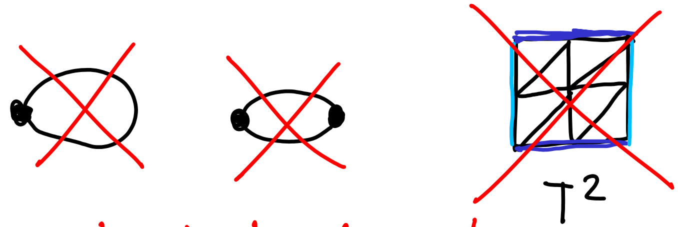
• sends vertices of Δ^k to $\coprod_{v \in F} \Delta^0 \times \{\{v\}\}$



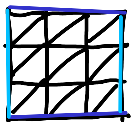
$S^1 \cong$ X^1

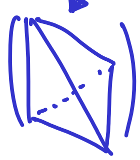
$D^2 \cong$ X^2

X^0

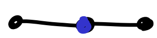


not simplicial complexes

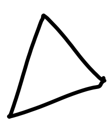
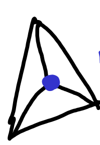
 T^2

$S^2 = \partial$  Δ^3

Barycentric subdivision "refines" the simplicial complex by adding more cells, while still producing the same polyhedron / homeo.

Δ^1  $\rightsquigarrow$  two Δ^1's

$V_{barycenter}$

 $\rightsquigarrow$  $V_{barycenter}$

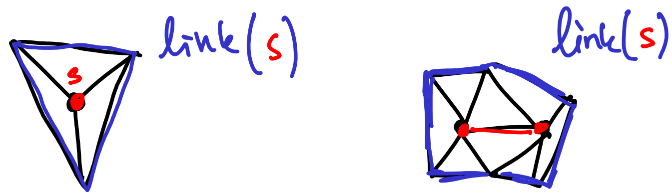
Δ^2 three Δ^2's

- Add an auxiliary vertex $V_{barycenter} := \left(\frac{1}{n+1}, \dots, \frac{1}{n+1}\right) \in \Delta^n$
- Divide Δ^n into the $n+1$ different convex hulls spanned by $\{V_{bary}, V_1, \dots, V_n\} \subseteq \Delta^n$, V_i distinct vertices of Δ^n

<u>Facts</u>

- Topological manifolds are not necessarily polyhedra in dim ≥ 4 (dim $= 4$: Casson-Freedman '80s, dim > 4 Manolescu '13)

- Smooth manifolds are polyhedra by [Whitney '40s], (such that the link of every simplex <u>moreover</u> is a sphere)



We now continue the proof of Prop. 6.

Fix $[\alpha] \in \pi_k(S^n, N)$. Consider a triangulation of S^k which is sufficiently fine so that

$\|\alpha(x) - \alpha(y)\| < \varepsilon$ (Use unif. cont.)

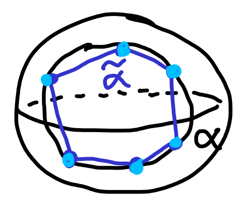
when x, y are contained in the same simplex.

Extend the restriction $\alpha|_{\text{vertices}}$ to a cont. map $\bar{\alpha}: S^k \to \mathbb{R}^{n+1}$ which is linear on each simplex (piecewise lin. on S^k).

(restriction to $\Delta^k \subseteq \mathbb{R}^{k+1}$ of a linear, rather)

For $\varepsilon > 0$ it is the case that:

- $\tilde{\alpha} := \bar{\alpha}/\|\bar{\alpha}\| : S^k \to S^n$ well def. & cont. $\tilde{\alpha}(N) = N$ (after a minor tweak)
- $[\tilde{\alpha}] = [\alpha]$ (use Lemma 7)



Case $1 \leq k < n$: $\tilde{\alpha}$ misses some $x \in S^n \setminus \{N\}$ [Why?]

Since $S^n \setminus \{x\} \cong \mathbb{R}^n$, we can simply contract $\tilde{\alpha}$ onto N:

$$\underset{\substack{\cap \\ \mathbb{R}^n = S^k \setminus \{x\}}}{(1-t)\tilde{\alpha} + tN} : S^k \times \underset{\substack{\Downarrow \\ t}}{[0,1]} \longrightarrow \mathbb{R}^n = S^n \setminus \{x\}$$

Case $k=n>0$: As above: $\tilde{\alpha}\big|_{\bigcup (n-1)\text{-simpl.}}$ misses some $x \in S^n \setminus \{N\}$.

After an initial generic perturbation of $\alpha\big|_{\text{vertices}}$ we may assume that $\tilde{\alpha}\big|_{\sigma^n}$ is the proj. to S^n of a (restriction to Δ^n of a) lin. map of full rank for any n-simplex σ^n.

$\Rightarrow$ $\tilde{\alpha}$ hits x at most once, and in the interior of each n-simplex σ^n.

Construct cont. $\phi: S^n \to S^n$ s.t. (see prev. lecture)

- $[\phi] = [\mathrm{id}_{S^n}] \in \pi_n(S^n, N)$
- $\phi\big|_{B_\delta(x)} = \mathrm{id}_{S^n}$
- $\phi^{-1}(x) = x$
- $\phi(S^n \setminus B_{2\delta}(x)) = \{N\}$

$\delta > 0$ small

a map $\Delta^n \to S^n$, $\partial\Delta^n \to \{N\} \subseteq S^n$

$$\Rightarrow [\tilde{\alpha}] = [\phi \circ \tilde{\alpha}] = \prod_{\sigma\ n\text{-splx}} [\phi \circ \tilde{\alpha}|_\sigma]$$

"orientation reversing" case

Exercise 7.) Show that $[\phi \circ \tilde{\alpha}|_\sigma] = [\mathrm{id}_{S^n}], [\mathrm{refl}_{S^n}] = -[\mathrm{id}_{S^n}]$, or $[0]$.

$\Rightarrow \pi_n(S^n)$ is a cyclic group.

By Prop. 4 $(n=1)$ & Exc. 6 $(n>1)$:

$$\pi_k(S^n, N) = [S^k, S^n]$$

Consider the vector-field

$$\bar{F}(\bar{x}) = \frac{1}{\mathrm{Area}(S^n)\cdot\|\bar{x}\|^{1+n}}(x_1,\dots,x_{n+1}) \quad \text{on } \mathbb{R}^{n+1}\setminus\{0\}$$

i.e. $\oint_{S^n_r} \bar{F}\bullet\vec{n}\,dS = \mathrm{Area}(S^n_r)/\mathrm{Area}(S^n)\cdot r^n = 1 \quad \forall r>0$

$$\mathrm{div}\,\bar{F}(\bar{x}) = \frac{\partial F}{\partial x_1} + \dots + \frac{\partial F_{n+1}}{\partial_{n+1}} = 0 \quad (\text{away from } 0\in\mathbb{R}^{n+1})$$

Define $\mathrm{wind}: \pi_n(S^n, N) \to \mathbb{R}$ by:

Choose a <u>smooth</u> repr. of $[\alpha] \in [S^n, S^n]$ (piecewise lin. approx / smoothing kernel)

- <u>Gauß divergence thm</u>: ($\mathrm{div}\,\bar{F}=0$ away from $0\in\mathbb{R}^{n+1}$)
  $\mathrm{wind}(\alpha) = \oint_\alpha \bar{F}\bullet\vec{n}\,dS$ well-defined on $\pi_n(S^n, N)$
  & is a group homomorphism.
- $\mathrm{wind}(k\cdot[\mathrm{id}_{S^n}]) = k \in \mathbb{Z}$ by an explicit calculation.

$\Rightarrow$ wind is an iso. of groups $\pi_n(S^n, N) \xrightarrow{\cong} \mathbb{Z}$.

$\square$