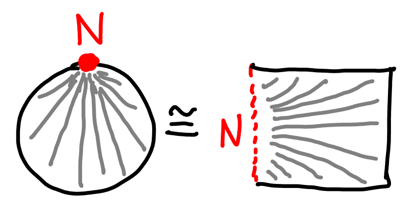
Proof of Thm. 8 (works for all loc. trivial fibrations)

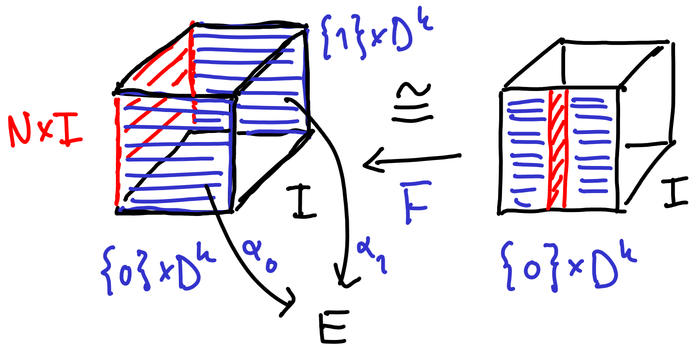
The crucial technique is the homotopy lifting property:

(1) any $\alpha : D^k \to B$ has a (non-unique!) lift
$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ D^k \arrow[r, "\alpha"'] \arrow[ur, "\tilde{\alpha}"] & B \end{tikzcd}$$

(2) any fixed lift β of the restriction $\alpha|_{\{0\}\times Y}$ of $\alpha : I \times Y \to B$ has a lift $\tilde{\alpha} : I \times Y \to E$ that satisfies $\tilde{\alpha}|_{\{0\}\times Y} = \beta$.

Rmk
- (1) is a special case of (2), since we always can lift $\alpha|_{\{N\}}$ $(N \in S^{k-1} = \partial D^k)$ and since
$$D^k \setminus \{N\} \cong (0,1] \times D^{k-1} \qquad (*)$$


(maps from $I \times D^{k-1}$ cst. on $\{0\} \times D^{k-1}$ $\leftrightarrow$ maps from D^k)

- If the two lifts $\tilde{\alpha}_i : D^k \to E$, $i = 0,1$, coincide at $N \in S^{k-1}$, then they are homotopic through lifts by (2) applied to
$$A := \alpha \circ pr_{D^k} \circ F : I \times D^k \to E$$


and the (partial) lift $\begin{cases} \tilde{\alpha}_i, & \text{on } \{i\}\times D^k \\ \tilde{\alpha}_i(N), & \text{on } I \times \{N\} \end{cases}$ precomposed w. F (again using $(*)$)

We prove (2) in the case when Y is compact & moreover has a triangulation w. finitely many simplices.

<u>Preparation</u>

Take some open cover $\{U\}$ of B w. loc. triv.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\Phi_U"] \arrow[dr, "\pi"'] & U \times G \arrow[d, "pr_U"] \\ & U \end{tikzcd}$$

Take a suff. fine triangulation of Y & suff. small number $\varepsilon = \frac{1}{L+1} > 0$ such that:

for each simplex $\Delta \subseteq Y$, & $\ell = 0, 1, \dots, L$,

$$\alpha\big([\ell \cdot \varepsilon, (\ell+1)\cdot\varepsilon] \times \Delta\big) \subseteq U$$

for some U in the cover

$$\boxed{\text{Choose one such } \Phi_{U_{(\Delta,\ell)}} \text{ for each } (\Delta, \ell)}$$

If we can extend $\tilde{\alpha}\big|_{\{0\}\times Y}$ to $\tilde{\alpha}\big|_{[0,\varepsilon]\times Y}$ we are done (induction on ℓ)

The latter extension is performed by induction on the skeleton.

<u>0-skeleton:</u> For each vertex $\Delta^0 \subseteq Y^0 \subseteq Y$ (singleton)

$$\Phi_{\mathcal{U}(\Delta^0,0)} \circ \tilde{\alpha}|_{\{0\}\times\Delta^0} : \{0\}\times\Delta^0 \to \{(\alpha(\Delta^0), g_{\Delta^0})\} \in \mathcal{U}\times G$$

$\vdots$

extend $\boxed{\text{constantly}}$ in the G-factor: $\Phi_{(\Delta^0,0)} \circ \tilde{\alpha}|_{[0,\varepsilon]} := \left(\alpha|_{[0,\varepsilon]\times\Delta^0}, \; g_{\Delta^0}\right)$

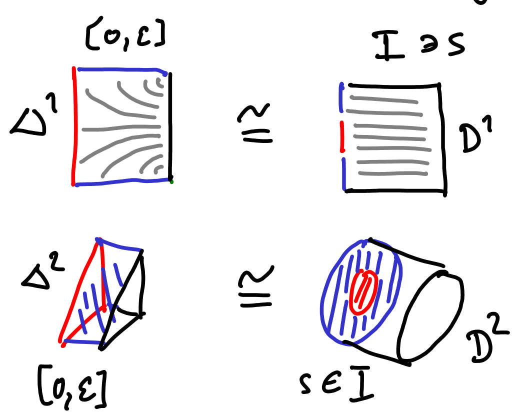
const!

⚠ makes sense only rel. <u>chosen</u> $\Phi_{(\Delta^0,0)}$

<u>k-skeleton</u> For any $\Delta^k \subseteq Y^k$, $\partial\Delta^k \subseteq Y^{k-1}$, $\tilde{\alpha}|_{[0,\varepsilon]\times\partial\Delta^k}$ was constructed in step $k-1$.

I.e. need to extend lift from $\{0\}\times\Delta^k \cup [0,\varepsilon]\times\partial\Delta^k$ to $[0,\varepsilon]\times\Delta^k$. Use e.g. the homeo:

$$\begin{array}{ccc} [0,\varepsilon]\times\Delta^k & \cong & I_s\times D^k \\ \cup & & \cup \\ ([0,\varepsilon]\times\partial\Delta^k)\cup(\{0\}\times\Delta^k) & \cong & \{0\}\times D^k \end{array}$$



and make the G-component of $\tilde{\alpha}$ <u>independent</u> of the coord $s \in I$ <u>relative</u> the chosen triv. $\Phi_{\mathcal{U}(\Delta^k,0)}$.

(i.e. the lift has cst. G-component along grey lines)

<u>Exercise 9.)</u> Check the exactness at all places!

□

Examples and computations

cartesian product of groups

Prop. 9 $\pi_i(X \times Y, (pt_X, pt_Y)) \cong \pi_i(X, pt_X) \times \pi_i(Y, pt_Y)$ (general fact!)

But in the nontrivial case Thm. 8 is crucial.

1.) The real projective space (space of unoriented lines through the origin)

$\mathbb{Z}_2 \circlearrowright \mathbb{R}^{n+1}$ mult. by $\{\pm 1\}$

$\cup$

$\mathbb{Z}_2 \circlearrowright S^n \rightsquigarrow \quad G \hookrightarrow E \twoheadrightarrow B$

(-1: antipodal map) $\quad \mathbb{Z}_2 \hookrightarrow S^n \twoheadrightarrow \mathbb{R}P^n \quad (\mathbb{R}P^1 = S^1)$

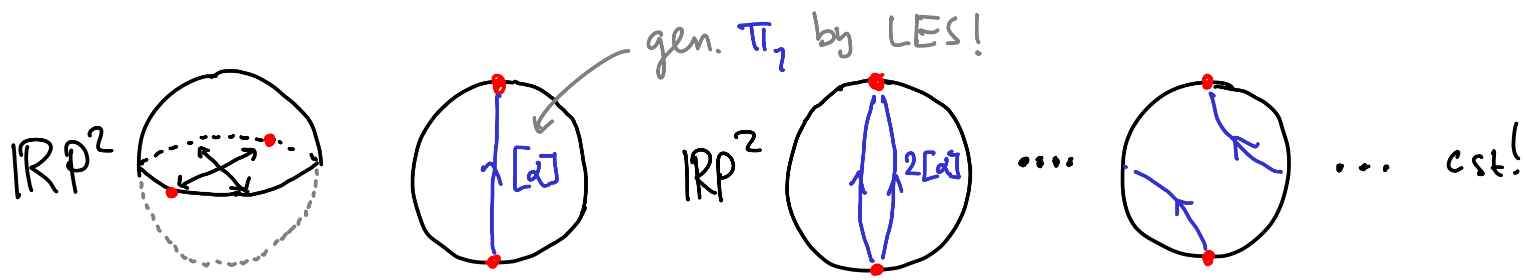
LES:

$$0 \to \pi_i(S^n) \xrightarrow{\cong} \pi_i(\mathbb{R}P^n) \to 0, \quad i > 1$$

$$0 \to \pi_1(S^n) \hookrightarrow \pi_1(\mathbb{R}P^n) \twoheadrightarrow \mathbb{Z}_2 \to 0$$

$n > 1$:

$$\boxed{\begin{aligned} &\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2, \quad n > 1 \\ &\pi_n(\mathbb{R}P^n) = \mathbb{Z}, \\ &\pi_i(\mathbb{R}P^n) = 0, \quad 1 < i < n \end{aligned}}$$



Facts:

- $S^n \cong \widetilde{\mathbb{R}P^n}$ above bundle is the univ. cover!
- $S^1 \subseteq S^2 \subseteq \dots \subseteq S^\infty \qquad \pi_i(S^\infty) = 0 \quad \forall i$

$\quad \downarrow \qquad \downarrow \qquad \qquad \downarrow$

$\quad \mathbb{R}P^1 \subseteq \mathbb{R}P^2 \subseteq \dots \subseteq \mathbb{R}P^\infty \qquad \pi_i(\mathbb{R}P^\infty) = 0 \quad i \neq 1$

$\pi_1(\mathbb{R}P^\infty) = \mathbb{Z}_2$

2.) The complex projective space (the space of cplx lines through the origin)

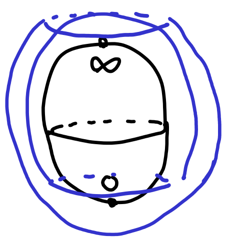
$G = U(1) = S^1$

$E = S^{2n+1} \subseteq \mathbb{C}^{n+1}$

$S^1 \hookrightarrow S^{2n+1} \twoheadrightarrow \mathbb{C}P^n$ "frame bundle of the tautological bundle"

LES $\Rightarrow$ $\boxed{\begin{aligned} &\pi_i(\mathbb{C}P^n) = 0, \quad i = 0, 1, 3, 4, \dots, 2n \\ &\pi_i(\mathbb{C}P^n) = \mathbb{Z}, \quad i = 2, 2n+1 \end{aligned}}$

$\mathbb{C} \cong V = \{[1 : z_2]\}$ $\quad$ $\underline{n=2}$: the Hopf bundle $S^1 \hookrightarrow S^3 \twoheadrightarrow S^2 = \mathbb{C}P^2$



$\mathbb{C}P^1 = \{[z_1 : z_2]\} \cong S^2$

transition function on $\mathbb{C}^* = U \cap V$:

$\mathbb{C} \cong U = \{[z_1 : 1]\}$ $\quad$ $\phi_{VU}(z_1) = e^{i \arg z_1} \in S^1$

(The section on U which is cst. w.r.t. the triv. Φ_U rotates one turn in the triv. Φ_V $\Rightarrow$ does not extend to "∞")

Facts

- $S^1 \to S^\infty \to \mathbb{C}P^\infty$ $\quad$ $\pi_i(\mathbb{C}P^\infty) = \begin{cases} 0, & i \neq 2 \\ \mathbb{Z}, & i = 2 \end{cases}$

- $S^1 \cong$ $S^1 / \langle e^{i 2\pi / k} \rangle \hookrightarrow S^{2n+1} / \langle e^{i 2\pi/k} \rangle \xrightarrow{\pi_k} \mathbb{C}P^n = S^{2n+1} / S^1$ $\quad$ tautological $\mathbb{C}$-bundle

  $\boxed{\text{cycl. subgp. of ord. } k}$

  $\rightsquigarrow \phi_{VU}(z_1) = e^{ik \arg z_1}$ $\quad$ $\boxed{\text{c.f. } \mathcal{O}(-k) = \mathcal{O}(-1)^{\otimes k}}$

- $\underline{\text{holomorphic category}}$: replace S^1, S^{2n+1} by $\mathbb{C}^*, \mathbb{C}^{n+1} \setminus \{0\}$

The following result allows us to concentrate on <u>compact</u> groups (as far as topology is concerned).

<u>Exercise 10.)</u> Use the Gram-Schmidt algorithm to show that $Gl_n(\mathbb{R}) \sim O(n)$ & $Gl_n(\mathbb{C}) \sim U(n)$.