The following result allows us to concentrate on <u>compact</u> groups (as far as topology is concerned).

<u>Fact:</u> $\underset{\frac{n\cdot(n-1)}{2}\text{ dim}}{O(n)}$, $\underset{(n+1)n-n=n^2\text{ dim}}{U(n)}$, $\overset{\det=1}{\underset{(n+1)\cdot n-n-1\text{ dim}}{SU(n)}}$ are compact <u>manifolds</u>.

$$SO(n) \hookrightarrow O(n) \xrightarrow{\det} \mathbb{Z}_2$$
$$SU(n) \hookrightarrow U(n) \xrightarrow{\det} S^1$$
principal bundles

<u>Exercise 11.)</u> Use the Gram-Schmidt algorithm to show that $Gl_n(\mathbb{R}) \sim O(n)$ & $Gl_n(\mathbb{C}) \sim U(n)$.

<u>Prop 10.</u> The universal cover $\tilde{G}_e \xrightarrow{\pi} G$ of a connected topological mfd. group (G, μ, e) has itself a unique (& natural) structure of a topological group, for which

$$1 \to \pi_1(G,e) \hookrightarrow \tilde{G}_e \xrightarrow{\pi} G \to 1 \qquad (*)$$

becomes a (short) <u>exact sequence</u> of groups.

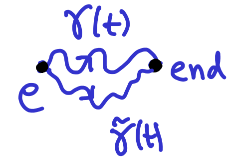
<u>Ex.</u>

$$0 \to \underset{\mathbb{Z}^n}{\underset{\shortparallel}{\pi_1(\mathbb{T}^n)}} \hookrightarrow \underset{\mathbb{R}^n}{\underset{\shortparallel}{\tilde{\mathbb{T}}^n_{(1,1,\dots,1)}}} \twoheadrightarrow \overset{n\text{ tours }=(O(1))^n}{\underset{(S^1)^n}{\underset{\shortparallel}{\mathbb{T}^n}}} \to 0$$

is a short exact sequence

<u>Proof:</u> Recall that $x \in \tilde{G}_e$ is a htpy. class of paths in G. We can define a product structure on such homotopy classes by

$x = [\gamma] = [\tilde{\gamma}]$



$$\tilde{\mu}\big([\gamma_0(t)],[\gamma_1(t)]\big) \overset{\text{def.}}{=} [\mu(\gamma_0(t),\gamma_1(t))] \quad (\text{well-def!})$$

<u>Exercise 12.)</u>* Derive uniqueness by showing that any cont. product making $(*)$ exact coincides w. $\tilde{\mu}$ on a <u>nonempty</u> <u>open</u> subset.

$\square$

* technical

More examples

$\begin{bmatrix} \boxed{*} & 0 \\ 0 & 1 \end{bmatrix}$ "the last row"

4.) $SO(n-1) \subseteq SO(n) \twoheadrightarrow S^{n-1}$.

LES $\Rightarrow$ $\boxed{\underline{n-1>i+1}: \ \pi_i(SO(n-1)) \xrightarrow{\cong} \pi_i(SO(n))}$

$SO(2) = S^1$

<u>Claim</u>. $SO(3) \cong \mathbb{RP}^3$

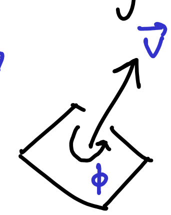
<u>Pf</u>. Recall: • $\mathbb{RP}^3 = S^3/\text{antipodal map} = \overbrace{S^3 \cap \{z \geq 0\}}^{\text{closed upper hemisphere}} / \text{antipodal on bdy } S^2$

$= D^3 / \text{antipodal on } \partial D^3 = S^2$

• Any $A \in SO(3)$, $A \neq \mathrm{Id}$, has an eigenspace of $\boxed{\dim = 1}$ corr. to the eigenvalue $\boxed{1.}$

We can describe A by: (1) choosing an orientation $\vec{v}$ of the eigenspace

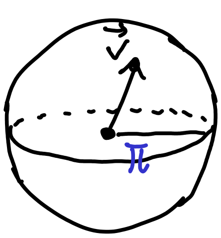
(2) performing a <u>pos.</u> rotation by $\phi \in (0, \pi]$ radians around $\vec{v}$



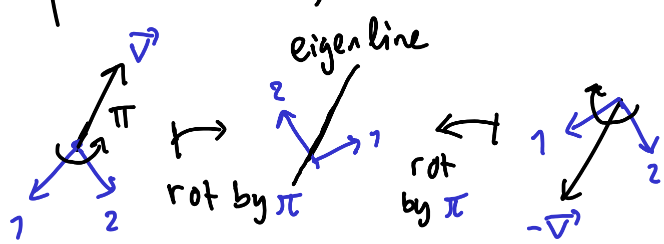
$\mathbb{RP}^3 \to SO(3)$

$0 \mapsto \mathrm{Id} \in SO(3)$

$v \mapsto$ rot by $\|v\| \in [0, \pi)$ radians around $v/\|v\|$ (pos. orientation!)



$D^3(\pi) / \text{antipod on } \partial D^3$ since:



$\square$

LES $\underline{n=3}$: $\pi_2(S^2) \to \underbrace{\pi_1(SO(2))}_{\mathbb{Z}} \twoheadrightarrow \underbrace{\pi_1(SO(3))}_{\mathbb{Z}_2} \to 0$ $\leftarrow \|$ c.f. Exc. (1.) from last lect.

$\mathbb{Z}_2 \hookrightarrow \boxed{Spin(n)} := \widetilde{SO(n)} \xrightarrow{\pi} SO(n)$ (univ. cover)

$\leftarrow$ a group by <u>Prop. 10</u>

$\underline{n=3}$: $S^3 \cong Spin(3) \cong SU(2) \leftarrow$ unit quaternions

$$\boxed{\begin{array}{l} a+bi+cj+dk \\ a^2+b^2+c^2+d^2=1 \\ ij=k,\ jk=i,\ ki=j \\ i,j,k \in SU(2) \end{array}}$$

action $SU(2) \circlearrowright \mathbb{R}^3$ induced by π:

conjugation on $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k \cong \mathbb{R}^3$

5.) $U(n-1) \subseteq U(n) \twoheadrightarrow S^{2n-1}$

$U(1) = S^1$

<u>LES</u> $\Rightarrow$ $\dfrac{\underline{n \geq 1}\ \&}{\underline{2n-1 > i+1}}$: $\boxed{\begin{array}{l} \pi_i(U(n-1)) \xrightarrow{\cong} \pi_i(U(n)) \\ \Rightarrow \pi_1(U(n)) = \mathbb{Z} \quad n \geq 1 \end{array}}$

6.) $SU(n) \hookrightarrow U(n) \twoheadrightarrow S^1$

<u>LES</u> $\Rightarrow$ $\boxed{\pi_i(SU(n)) \cong \pi_i(U(n)) \quad i > 1}$

$SU(2) \cong S^3 \Rightarrow \pi_2(U(n)) \cong \mathbb{Z} \quad n \geq 2$

$\pi_1(SU(n)) = 0$ since

<u>Exercise 13.)</u> $U(n) \xrightarrow{\det} S^1$ induces <u>iso.</u> $\pi_1(U(n)) \to \pi_1(S^1)$

<u>Hint:</u> surjectivity by hand, injectivity is shown by induction on n, use iso. in (5.) above.

7.) <u>Grassmanians</u> over either $\mathbb{R}$ or $\mathbb{C}$.

$$Gl_{n-m} \times Gl_m \subseteq Gl_n \twoheadrightarrow Gr_m(n)$$

$$\begin{bmatrix} \boxed{A} & 0 \\ 0 & \boxed{B} \end{bmatrix}$$

$(n-m)m$-dim cpct. mfd that parametrises the m-planes in $\mathbb{R}^n$ (or $\mathbb{C}^n$)

alternatively: $O(n-m) \times O(m) \subseteq O(n) \twoheadrightarrow Gr_m(n;\mathbb{R})$

$$U(n-m) \times U(m) \subseteq U(n) \twoheadrightarrow Gr_m(n;\mathbb{C})$$

For $n \gg 0$, m fixed, k fixed:

$$O(n-m) \subseteq O(n-m+1) \subseteq \dots \subseteq O(n)$$

$$\searrow \; \downarrow S^{n-m} \qquad \searrow \; \downarrow S^{n-1}$$

induces isomorphism on π_i for all $i \leq k$ (use Ex. (4.))

Similarly for $U(n-m) \subseteq \dots\dots \subseteq U(n)$ (use Ex. (5.))

<u>Hence</u> $O(n-m) \times O(m) / O(n-m) \times Id \hookrightarrow \underbrace{O(n)/O(n-m)} \twoheadrightarrow Gr_m(n;\mathbb{R})$

$\qquad \cong O(m)$

$\pi_i = 0 \; \forall n \gg 0$ (i fixed)

"space of m-frames in $\mathbb{R}^n$"

Similarly $U(m) \hookrightarrow \underbrace{U(n)/U(n-m)} \twoheadrightarrow Gr_m(n;\mathbb{C})$

$\pi_i = 0 \; \forall n \gg 0$ (i fixed) limit of spaces as $n \to +\infty$ yields principal bundles

$\boxed{\pi_i = 0 \quad \forall i}$

$$O(m) \hookrightarrow EO(m) \twoheadrightarrow BO(m) = Gr_m(\infty;\mathbb{R})$$
$$U(m) \hookrightarrow EU(m) \twoheadrightarrow BU(m) = Gr_m(\infty;\mathbb{C})$$

$\left(X_1 \subseteq X_2 \subseteq \dots,\ X_\infty = \bigcup_i X_i;\quad U \subseteq X_\infty \text{ open} \iff U \cap X_i \text{ open } \forall i \right)$

We have already seen the cases when $m = 1$:

$$\begin{array}{ccccc} \mathbb{Z}_2 & \hookrightarrow & S^\infty & \twoheadrightarrow & \mathbb{RP}^\infty \\ & & \| & & \| \\ & & EO(1) & & BO(1) \end{array}$$

$$\begin{array}{ccccc} S^1 & \hookrightarrow & S^\infty & \twoheadrightarrow & \mathbb{CP}^\infty \\ & & \| & & \| \\ & & EU(1) & & BU(1) \end{array}$$

Later we will see that these are the "universal" principal $O(m)$ or $U(m)$ bundles.

Analogously: $Gl_m \hookrightarrow EG(m) \twoheadrightarrow BG(m)$

For a closed subgroup $G \subseteq Gl_m$ (e.g. discrete)

$$\boxed{\begin{array}{ccccc} G & \hookrightarrow & EG & \twoheadrightarrow & EG(m)/G =: BG \\ & & \| & & \\ & & EG(m) & & \end{array}}$$

the <u>classifying space</u> of G
- unique/htpy.
- $H^n(BG;\mathbb{Z}) =$ gp. cohom.

(alt: $BG = BO(m)/G,\ BU(m)/G$ if $G \subseteq O(m)$, etc.)

Ex

- $B\mathbb{Z}_2 = \mathbb{R}P^\infty$
- $BS^1 = \mathbb{C}P^\infty$
- $B\mathbb{Z} \simeq S^1$