The Poincaré homology 3-sphere

the symmetric group

$S_n \leq O(n)$ is the gp. of order $n!$ which preserves $\Delta^{n-1} \subseteq \mathbb{R}^n$.

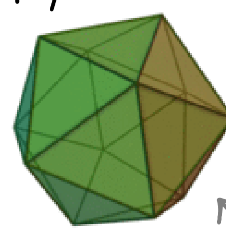
$A_n := S_n \cap SO(n)$ normal subgroup of order $n!/2$.

the alternating group

Facts

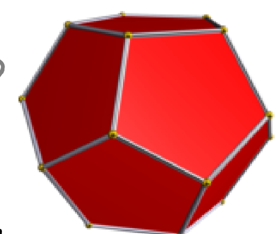
- $A_5 / [A_5, A_5] \cong \{1\}$ ("perfect" group)
- $A_5 \hookrightarrow SO(3)$ is the symmetry group of the regular icosahedron.
  20

  or of the regular dodekahedron
  12



20 faces

12 faces



- $SO(3)/A_5$ is a closed 3-dim manifold called the Poincaré homology sphere.

To compute $\pi_1(SO(3)/A_5)$ we go to the universal cover

$$\begin{array}{ccccc} \mathbb{Z}_2 & \hookrightarrow & SU(2) = S^3 & \twoheadrightarrow & SO(3) \\ \cup\!\mid & & \cup\!\mid & & \cup\!\mid \\ \mathbb{Z}_2 & \hookrightarrow & I & \twoheadrightarrow & A_5 \end{array}$$

binary icosahedral gp.
order $= 2 \cdot |A_5| = 120$

- I is again perfect

  $I = \langle r, s, t \mid r^2 = s^3 = t^5 = rst \rangle$

- Thus $S^3/I = SO(3)/A_5$

the Poincaré homology 3-sphere $H_1 = 0$, $\pi_1 = I$)

<u>LES</u>: $I \hookrightarrow S^3 \twoheadrightarrow S^3/I$

$$0 = \pi_1(S^3) \to \pi_1(S^3/I) \xrightarrow{\cong} \pi_0(I) = I \to 0$$

<u>So</u>: $H_1 = 0$ does not characterise S^3

(but $\pi_1 = 0$ does by Perelman's result)

Morphisms & classification

Recall that a morphism of principal G-bundles is nothing but a cont. G-equivariant map

$$\begin{tikzcd} E_1 \arrow[r, "\Psi"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & E_2 \arrow[d] \\ B_1 = E_1/G \arrow[r, "\psi"'] & E_2/G = B_2 \end{tikzcd}$$

orbit space orbit space

$\Psi(x)\cdot g = \Psi(x\cdot g) \quad \forall\, g \in G.$

$\Rightarrow$ • bijective on each G-orbit $E \supseteq x\cdot G = \pi^{-1}(\pi(x)) \cong G$

Prop. 11. Any morphism of G-bundles that covers a homeomorphism is itself invertible w. an equivariant inverse.

Proof. Bijective since bijective on each fibre (orbit) & since $\psi : B_1 \to B_2$ is an bijection on the orbit space.

Openness can be checked in a local trivialisation. $\square$

Def The gauge transformations $\mathcal{G}(E)$ of a principal bundle $E \twoheadrightarrow B$ is the $\boxed{\text{group}}$ of equivariant maps that cover $\psi = \mathrm{id} : B \to B$. By Prop. 11.

Ex • $E = B \times G$ (trivial bundle) then $\mathcal{G} = C(B, G)$

$$x = (b,h) \mapsto (b, f(b)\cdot h) \quad , \quad f \in C(B,G)$$

[clearly equivariant: $(b,h)\cdot g \mapsto (b, f(b)\cdot h)\cdot g$]

• $Z(G)$ centre (elements that commute w. everything)

$C(B, Z(G)) \subseteq \mathcal{G}(E)$ $\quad x \mapsto x \cdot f(b)$

[$x\cdot g \mapsto x\cdot g \cdot f(b) = (x \cdot f(b))\cdot g$]

Pullback Given a cont. map $\psi : B' \longrightarrow B$, and a principal G-bundle $E \overset{\pi}{\twoheadrightarrow} B$, there exists a (unique up to iso.) G-bundle $\psi^* E \xrightarrow{\pi'} B'$ which admits a morphism $\psi^* : \psi^* E \longrightarrow E$ that covers ψ, i.e.

$$\begin{tikzcd}
\psi^* E \arrow[r, "\psi^*"] \arrow[d, two heads, "\pi'"'] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[d, two heads, "\pi"] \\
B' \arrow[r, "\psi"'] & B
\end{tikzcd}$$

ψ^* morphism of G-bundles.

$$\psi^* E := B' \underset{\psi,\pi}{\times} E := \{(x,y) \mid \psi(x) = \pi(y)\} \subseteq B' \times E \circlearrowleft \begin{array}{l} G \text{ acts} \\ \text{on } E \end{array}$$

$$\pi' := pr_{B'}\big|_{\psi^* E} \left(\text{"="} \, id_{B'} \times \pi \big|_{\psi^* E} \right)$$

$$\begin{array}{ccc} \downarrow id_{B'}\times\pi & & \downarrow pr_{B'} \\ \Gamma_\psi & \xleftarrow{\ \cong\ } & B' \\ (b, \psi(b)) & \longleftarrow\!\shortmid & b \end{array}$$

$\psi^* := pr_E\big|_{\psi^* E}$ (G-equivariant)

Exercise 14.) 1.) Verify that $\psi^* E$ is a principal G-bundle (exhibit local trivialisations).

2.) Show that E trivial $\Rightarrow$ $\psi^* E$ trivial. In particular: ψ const. $\Rightarrow$ $\psi^* E$ trivial

3.) $\psi = \mathrm{id}: B \to B \Rightarrow \psi^* E = E$

Thm 12. If $E \twoheadrightarrow B \times [0,1]$ is a principal G-bundle, B cpct, then $\underset{B\times\{0\}}{\overset{\pi\downarrow}{E_0}} := \underbrace{\pi^{-1}(B \times \{0\})}_{= \iota^*_{B\times\{0\}} E}$ is isomorphic to $\underset{B\times\{1\}}{\overset{\pi\downarrow}{E_1}} = \underbrace{\pi^{-1}(B \times \{1\})}_{= \iota^*_{B\times\{1\}} E}$.

For principal bundles over a contractible (e.g. D^n) base B
$\left(\exists\ F: B \times I \to B :\ F(x,0) = x,\ F(x,1) \equiv x_0 \right)$
we now deduce:

Cor 13. Any principal G-bundle with a contractible base B is isomorphic to the trivial bundle $B \times G$.

Proof. Apply Thm. 11 to $F^* E$ & use Exc. (14.) $\square$

Proof of Thm 12. Using Prop 11, it suffices to construct an equivariant map

$$\begin{tikzcd} E_0 \arrow[r, "\Psi"] \arrow[d] & E_1 \arrow[d] \\ B\times\{0\} \arrow[r, "id"'] & B\times\{1\} \end{tikzcd}$$

Since there is an obvious inclusion $E_0 \subseteq E$, it suffices to construct an equivariant map

$$\begin{tikzcd} E \arrow[r, "\Psi"] \arrow[d] & E_1 \arrow[d] \\ B\times I \arrow[r, two heads, "pr_B"'] & B\times\{1\} \end{tikzcd}$$

For simplicity, let's assume that $E_1 = B\times\{1\}\times G$ is trivial.
By the homotopy lifting property (c.f. Proof of LES (Thm. 8)) any section $B\times\{1\} \xrightarrow{\sigma} E_1$ extends to a section

$$\begin{tikzcd} B\times I \arrow[r, "\Sigma"] \arrow[dr, equal] & E \arrow[d] \\ & B\times I \end{tikzcd}$$

∃ by triviality of E_1!

But $\boxed{\exists \text{ global section} \iff \text{trivial bundle}}$

In general, if

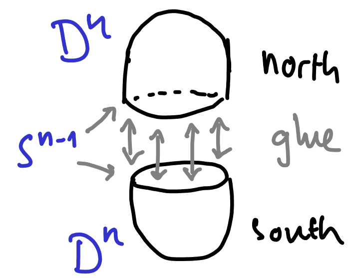
$$\begin{tikzcd} \tilde{B} \arrow[r, "\tilde{\Sigma}"] \arrow[dr, equal] & \tilde{E} \arrow[d, "\pi"] \\ & \tilde{B} \end{tikzcd}$$

is a section, then

$$\begin{aligned} \tilde{B}\times G &\longrightarrow \tilde{E} \\ (b,g) &\longmapsto \tilde{\Sigma}\cdot g \end{aligned}$$

is an isomorphism of G-bundles. $\square$

Classification



A sphere can be decomposed as

$$S^n = D^n_{\text{north}} \amalg D^n_{\text{south}} / \sim$$

i.e. two hemispheres glued along their boundary S^{n-1} (equator).

For any $G \hookrightarrow E \twoheadrightarrow S^n$:

$$(\text{Thm } 12.) \Rightarrow \quad E\big|_{\text{hemisphere}} \cong \text{hemisphere} \times G$$

Hence: E is obtained by gluing two trivial bundles on D^n along the boundary S^{n-1} using a possibly nontrivial identification $S^{n-1} \times G \longrightarrow S^{n-1} \times G$.

Recall: the gp. of gauge transform. of $S^{n-1} \times G$ is

$$\mathcal{G} = C(S^{n-1}, G)$$

(Thm 12.) $\Rightarrow$ homotopic maps give isom. bundles, ... $\Rightarrow$

Claim $\{\text{Principal } G\text{-bundles over } S^n / \text{iso.}\} \cong \pi_{n-1}(G) = [S^{n-1}, G]$

$$E \longmapsto \alpha_E$$

Using $G \hookrightarrow EG \twoheadrightarrow BG$ ← classifying space

$\pi_{n-1} = 0 = \pi_n$ (of EG)

LES $\Rightarrow$ $0 \to \pi_n(BG) \xrightarrow[\delta]{\cong} \pi_{n-1}(G) \to 0 \Rightarrow \pi_{n-1}(G) \cong \pi_n(BG)$.

The corr. $\{G\text{-bundles over } S^n/\text{iso}\} \cong \pi_n(BG)$ is natural:

Exercise 15.)* $\quad E \cong (\delta^{-1}(\alpha_E))^* EG$

The general result:

Thm 14. $[B, BG] = \{G\text{-bundles over } B/\text{iso.}\}$

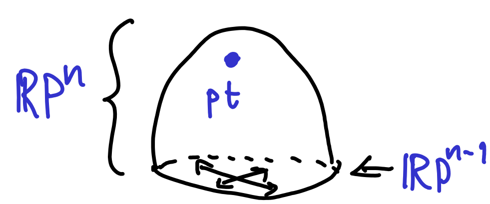
$$\varphi \longmapsto \varphi^* EG$$

Exercise 16.) Classify the principal G-bundles over B when:

- $B = S^n$ & $G = S^0 = \mathbb{Z}_2$
- $B = S^n$ & $G = S^1$
- * $B = \mathbb{R}P^n$ & $G = S^0 = \mathbb{Z}_2$ (Answer $= 2$)
- * $B = \mathbb{C}P^n$ & $G = S^1$ (Answer $= \mathbb{Z}$)

Hint: Start w. $n=1$

Then argue by induction, use



$\mathbb{R}P^n \setminus \{pt\} \sim \mathbb{R}P^{n-1}$, $\mathbb{C}P^n \setminus \{pt\} \sim \mathbb{C}P^{n-1}$

$\mathbb{R}P^n = D^n/\sim$ antipodal on S^{n-1}