IV Knot theory

Rough idea: classify embeddings $Y \hookrightarrow X$ up to $\mathrm{Aut}(X)$.
Since we will work in the smooth category, we first need some background on smooth manifolds.

1. Smooth manifolds

A smooth (C^∞) map $f: U \to V$, where $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ are open, is an

- <u>immersion at $x_0 \in U$</u> if $D_{x_0} f$ ($\in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m)$) is injective ($\Rightarrow m \geq n$)
- <u>submersion at $x_0 \in U$</u> if $D_{x_0} f$ is surjective ($\Rightarrow m \leq n$)

(f immersion/submersion if prop. holds at all points)

- <u>diffeomorphism</u> if f bijective & f^{-1} smooth

  (chain rule) $\Rightarrow f$ immersion & submersion $\Rightarrow m = n$

A diffeomorphism is a "smooth coordinate change".

<u><u>Non-Ex</u></u> $x \mapsto x^3$ is smooth, bijective, but <u>not</u> a diffeo.

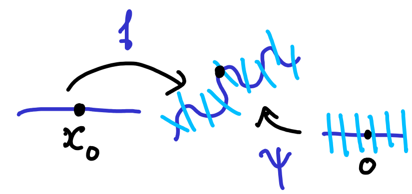
Thm. 15 (Implicit function theorem) Let $f: U \to \mathbb{R}^m$ be a smooth function, fix $x_0 \in U$, and write $y_0 := f(x_0)$. If f is

- a <u>submersion</u> at x_0, then $\exists$ neighbourhoods $U_0 \subseteq U$ of x_0 and $V_0 \subseteq \mathbb{R}^m$ of $f(x_0)$, diffeomorphisms $\varphi: \mathbb{R}^n \to U_0$, $0 \mapsto x_0$, $\psi: \mathbb{R}^m \to V_0$, $0 \mapsto y_0$

  s.t. $\boxed{\psi^{-1} \circ f \circ \varphi(x_1, \dots, x_n) = (x_1, \dots, x_m)}$ $(m \leq n)$.

- an <u>immersion</u> at x_0, then similarly $\exists$ nbhds & diffeo's

  s.t. $\boxed{\psi^{-1} \circ f \circ \varphi(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots, 0).}$

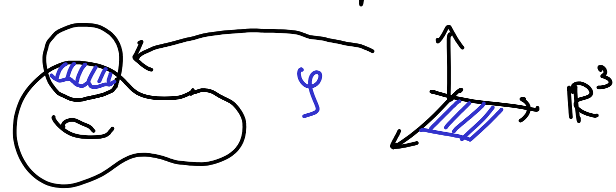


<u>Obs:</u> in particular if $f: U \to V$ smooth, bij., $D_x f \in GL_n$ (imm. & subm.)

$\Rightarrow$ f is locally a diffeomorphism (e.g. the k-fold cover $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^k$)

<u>Def</u> • An <u>embedded submanifold</u> (of $\mathbb{R}^n$) is a subset $M \subseteq \mathbb{R}^n$ for which every $p \in M$ has a nbhd. $U \subseteq \mathbb{R}^n$ s.t. $\exists$ diffeomorphism $\varphi: \mathbb{R}^n \to U$ where $\varphi^{-1}(M \cap U) = \mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^n$.

• A map $f: M \to N$ between embedded submanifolds is called <u>smooth</u> if it is the restriction of some smooth map defined on an open nbhd. of $M \subseteq \mathbb{R}^n$.



The restriction $\varphi|_{\mathbb{R}^m \times \{0\}} : \mathbb{R}^m \hookrightarrow M$ is called a

<u>local parametrisation</u>, which means its an injective immersion

with image being an open subset of M.

<u>In particular:</u> smooth submanifolds are topological manifolds.
(metric topology & 2nd countability inherited from $\mathbb{R}^n$)

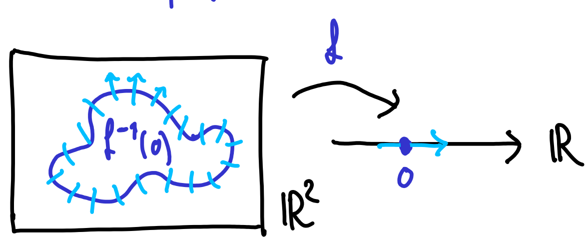
A smooth map $f: M \to N$ is called an immersion/submersion

if the same is true for the maps

$$\varphi_N^{-1} \circ f \circ \varphi_M : \mathbb{R}^{\dim M} \to \mathbb{R}^{\dim N}$$

expressed w.r.t. local parametrisations.

<u>Excercise 17.)</u> Show that the preimage $f^{-1}(pt) \subseteq \mathbb{R}^n$ of any smooth map $f: \mathbb{R}^n \to \mathbb{R}^m$ which is a submersion at all points in $f^{-1}(pt)$ is a submanifold of dimension $n-m$.



<u>Ex</u>

- $S^n \subseteq \mathbb{R}^{n+1}$ smooth n-dim submfd.
- $S^{n_1} \times S^{n_2} \times \ldots \times S^{n_m} \subseteq \mathbb{R}^{n_1 + \ldots + n_m + m}$

A smooth map $f: M \to N$ with a smooth inverse is called a <u>diffeomorphism.</u>

⚠ The map $f: M \to N$ is not necessarily the restriction of a global diffeomorphism $\begin{matrix} \mathbb{R}^n & \to & \mathbb{R}^m \\ \cup\!\! \mathsf{I} & & \cup\!\! \mathsf{I} \\ M & & N \end{matrix}$

<u>E.g.</u> $S^n \times \{0\} \subseteq \mathbb{R}^{n+2}$ is diffeom. to $S^n \subseteq \mathbb{R}^{n+1}$

<u>Prop 16.</u> If M is a compact (sub)manifold and $f: M \hookrightarrow \mathbb{R}^n$ is an injective smooth immersion, then the image $f(M) \subseteq \mathbb{R}^n$ is a smooth (sub)manifold and $f: M \to f(M)$ is a diffeomorphism.

<u>Proof</u> The local parametrisations of M give rise to local parametrisations $f \circ \varphi : \mathbb{R}^{\dim M} \hookrightarrow f(M) \subseteq \mathbb{R}^n$ of the image.

<u>Goal:</u> extend $f \circ \varphi$ to a diffeomorphism

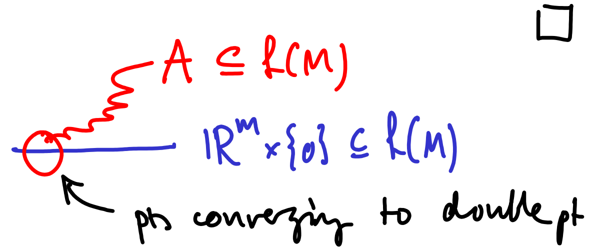
$$\Phi : \mathbb{R}^{\dim M} \times \mathbb{R}^{n - \dim M} \longrightarrow U, \quad \Phi^{-1}(f(M)) \overset{(*)}{=} \mathbb{R}^{\dim M} \times \{0\}$$

M cpct $\Rightarrow$ $f(M)$ cpct & $f|_M : M \to f(M)$ homeo! (gen. facts)
smoothness of f^{-1} can be seen in loc. coord's.

The sought extensions Φ exist as a consequence of the implicit function theorem (Thm. 15).

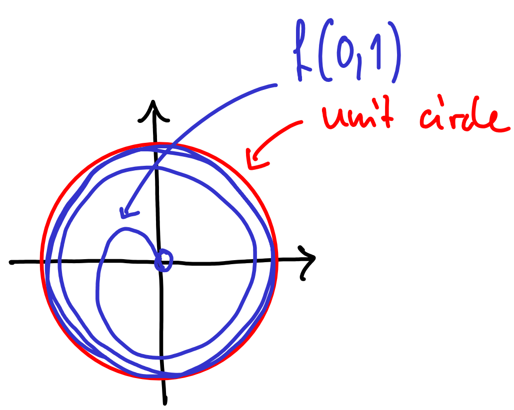
For (*) we need to use compactness. Assume the produced Φ satisfy $\Phi^{-1}(f(M)) = \mathbb{R}^{\dim M} \times \{0\} \cup A$, $A \neq \emptyset$.

- $A \subseteq \mathbb{R}^n \setminus \mathbb{R}^{\dim M} \times \{0\}$ closed $\rightsquigarrow$ shrink domain & target, done!
- otherwise, use compactness of M to show that f is not injective $\Rightarrow$ contradiction $\lightning$. $\square$



Non-Ex $f : \underbrace{S^1 \amalg (0,1)}_{\text{non-compact 1-dim mfd.}} \hookrightarrow \mathbb{R}^2$ which is the canonical inclusion on the S^1 factor, while $f|_{(0,1)}(x) = x \cdot e^{i \frac{1}{1-x}}$

Image of f:



conn., comp.

A **(one-dim) knot** is closed one-dim submanifold

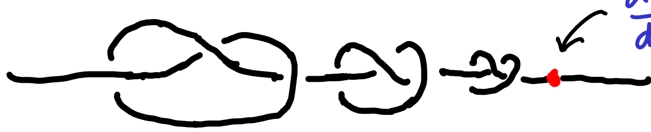
$K \subseteq \mathbb{R}^3$ (m nr. of) (m component)

A disjoint union of knots is called a **link**.

A knot can be obtained as the **image** of a smooth inj. immersion

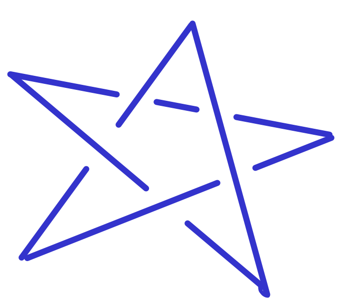
$$\gamma : \underset{\substack{\parallel \\ S^1}}{\mathbb{R}/2\pi\mathbb{Z}} \hookrightarrow \mathbb{R}^3 , \qquad \frac{d\gamma}{d\theta} \neq 0 \quad \text{(immersion)}$$

immersion is needed to rule out "wild knots"



(smooth)

Alternative model: Piecewise linear embeddings



(up to suitable equivalence)

Isotopies

In smooth knot theory one is typically interested in (smooth) submanifolds up to (smooth) <u>isotopy</u>, i.e.

$$\varphi : \underbrace{[0,1]\times M}_{\text{manifold}} \longrightarrow \mathbb{R}^n \quad \underline{\text{smooth}} \qquad (M \text{ compact})$$

$$\varphi_t = \varphi(t,\cdot) : \{t\}\times M \hookrightarrow \mathbb{R}^n \quad \underline{\text{injective immersion}} \quad \forall\, t\in[0,1]$$

(Prop. 16) $\Rightarrow$ $\varphi_t(M)$ "path" of submanifolds

Smoothness in the t-variable is <u>crucial</u>

shrink by lin. rescaling

Cont. but <u>cannot</u> be smooth w.r.t. t (nontrivial fact)

$t=0$ $\quad$ $t=1/2$ $\quad$ $t=1$

<u>Thm 17.</u> (Isotopy extension) Any smooth isotopy $\varphi_t : M \hookrightarrow \mathbb{R}^n$ of a compact manifold is of the form $\varphi_t = \Phi_t \circ \varphi_0$ for a compactly supported isotopy of the ambient $\mathbb{R}^n$, i.e. $\Phi_t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ diffeo's smoothly dep. on $t\in[0,1]$,

$$\Phi_0 = \mathrm{id}_{\mathbb{R}^n}$$

$$\Phi_t = \mathrm{id}_{\mathbb{R}^n} \quad \text{outside a fixed compact subset}$$