Recall that a function $f: M \to \mathbb{R}^m$ is smooth if it is the restriction of a smooth function defined on some open nbhd of $M \subseteq \mathbb{R}^n$ by our definition. The following <u>local</u> characterisation is crucial (& used implicitly in the proof of Prop. 16)

<u>Lem. 18</u> Let $M \subseteq \mathbb{R}^n$ be a submanifold. Any function $f: M \to \mathbb{R}^N$ which satisfies the property that $f \circ \varphi : \mathbb{R}^{\dim M} \to \mathbb{R}^N$ is smooth for any local parametrisation $\varphi : \mathbb{R}^{\dim M} \hookrightarrow M \subseteq \mathbb{R}^n$ is the restriction of a smooth function $F: \mathbb{R}^n \to \mathbb{R}^N$.

non-unique!

<u>Proof</u> Since φ can be extended to a diffeomorphism

$$\Phi : \underset{\bar{x}_1}{\mathbb{R}^{\dim M}} \times \underset{\bar{x}_2}{\mathbb{R}^{n-\dim M}} \hookrightarrow U \subseteq \mathbb{R}^n, \quad \Phi^{-1}(M \cap U) = \mathbb{R}^{\dim M} \times \{0\}$$

such an extension can be found in U, e.g. $F_U(\bar{x}_1, \bar{x}_2) = f(\bar{x}_1)$.

In this manner we may assume that we have a covering $\{U_i\}$ of $\mathbb{R}^n$ by open subsets, together w. smooth functions $F_i : U_i \to \mathbb{R}^N$ that satisfy $F_i|_{M \cap U_i} = f$.

These functions can now be patched together to form the globally def. function $F: \mathbb{R}^n \to \mathbb{R}^N$ by the following "partition of unity" argument:

1.) "Refine" the cover by constructing a cover by open balls $\{B_{r_i}(p_i)\}$, $r_i > 0$, $p_i \in \mathbb{R}^n$, s.t.

- $\overline{B_{r_i}(p_i)} \subseteq U_{\alpha(i)}$ and

"paracompactness"

which is <u>locally finite</u>, i.e. for each fixed i:

- $B_{r_i}(p_i) \cap B_{r_j}(p_j) = \emptyset$ for all but <u>finitely</u> many j.

2.) Consider smooth functions $\sigma_i : \mathbb{R}^n \longrightarrow [0, +\infty)$ which satisfy $\sigma_i^{-1}(0) = \mathbb{R}^n \setminus B_{r_i}(p_i)$.

3.) Take $F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ to be

$$F(\bar{x}) = \frac{\sum_i \sigma_i(\bar{x}) \cdot F_{\alpha(i)}(\bar{x})}{\sum_i \sigma_i(\bar{x})}$$

sums are <u>infinite</u>, but still make sense by (1.)

$\longleftarrow$ non-zero term $\exists$ by (2.)

<u>Obs</u>: $\bar{x} \in M \Rightarrow F(\bar{x}) = \dfrac{\sum_i \sigma_i(\bar{x}) \cdot f(\bar{x})}{\sum_i \sigma_i(\bar{x})} = f(\bar{x})$ $\square$

Proof (Isotopy extension theorem)

For convenience, assume $\varphi : \mathbb{R}_t \times M \longrightarrow \mathbb{R}^n$.

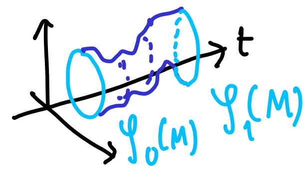
Prop. 16 $\Rightarrow$ $\varphi_t(M) \subseteq \mathbb{R}^n$ submanifold for each t

& $\varphi_t : M \longrightarrow \varphi_t(M)$ diffeomorphism

check by hand

$\Rightarrow$ $\mathrm{image}\big((t,\varphi_t) : \mathbb{R}\times M \to \mathbb{R}\times\mathbb{R}^n\big) = \{(t,x);\ x = \varphi_t(y),\ y \in M\}$ is a submanifold

↑ called the trace of φ_t

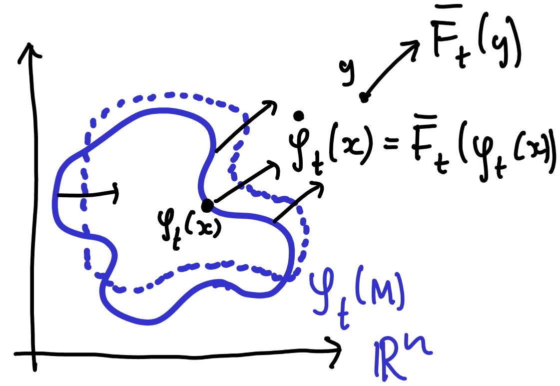


& $(t,\varphi_t) : \mathbb{R}\times M \longrightarrow \mathbb{R}\times\mathbb{R}^n$ is a diffeomorphism.

Hence, the smooth function

$$\underset{\underset{\text{submfd.}}{\cup\!\!\!\mid}}{\mathbb{R}\times\mathbb{R}^m}$$

$$\frac{\partial}{\partial t}\varphi : \mathbb{R}_t \times M \longrightarrow \mathbb{R}^n$$



gives rise to the smooth function

$$\frac{\partial}{\partial t}\varphi \circ (t,\varphi_t)^{-1} : \underset{\subseteq\, \mathbb{R}\times\mathbb{R}^n}{\text{trace}} \longrightarrow \mathbb{R}^n$$

Basically, Lem. 18 is responsible for ensuring that $(t,\varphi_t)^{-1}$ has a smooth extension.

In particular, $\exists$ smooth extension F to $\mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}$

i.e. a t-dependent vector field on $\mathbb{R}^n$ which may be assumed to vanish outside of some compact subset.

smooth in "t" as well

We write $\bar{F}_t : \mathbb{R}^n \to \mathbb{R}^n$, which by construction satisfies

$$(*) \quad \boxed{\frac{\partial}{\partial t} \varphi_t(x_0) = \bar{F}_t(\varphi_t(x_0)) \quad \text{for all } x_0 \in M.}$$

The existence of solution $\Phi_t(y) \in \mathbb{R}^n$ to the ODE

$$\begin{cases} \frac{d}{dt} \Phi_t(y) = \bar{F}_t(\Phi_t(y)) & \text{(ODE)} \\ \Phi_0(y) = y & \text{(IC)} \end{cases}$$

together with smooth dependence on the initial conditions (IC) gives the sought ambient isotopy

$$\Phi : \underset{\substack{\Psi \\ t}}{[0,1]} \times \underset{\substack{\Psi \\ y}}{\mathbb{R}^n} \longrightarrow \mathbb{R}^n \quad \text{smooth}$$

Indeed, uniqueness & $(*)$ $\Rightarrow$ $\Phi_t \circ \varphi_0 = \varphi_t$

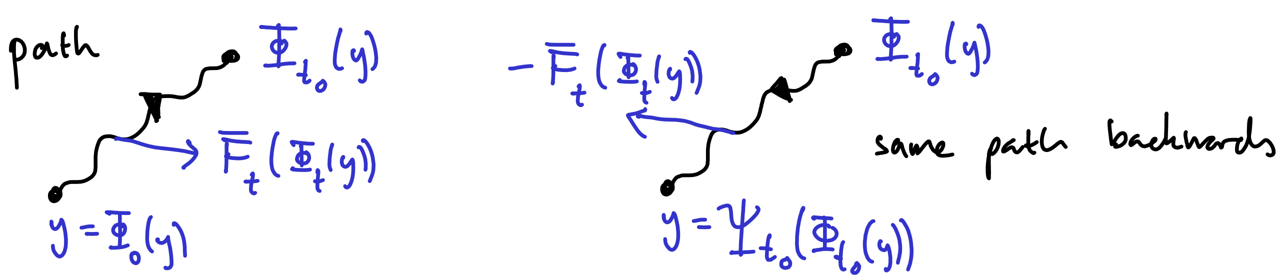
- Each Φ_t is a compactly supported since

  $\Phi_0 = id_{\mathbb{R}^n}$ & $\frac{d}{dt}\Phi_t = 0$ outside a cpct subset.

- Φ_{t_0} is a diffeomorphism for all $t_0 \in [0,1]$ since the solution $\Psi_t(y)$ of the "backwards flow":

$$\begin{cases} \frac{d}{dt}\Psi_t(y) = -F_{t_0 - t}(\Psi_t(y)) \\ \Psi_0(y) = y \end{cases}$$

satisfies $\Psi_{t_0} \circ \Phi_{t_0} = Id_{\mathbb{R}^n}$ by uniqueness of sol$\underset{\cdot}{\overset{s}{-}}$



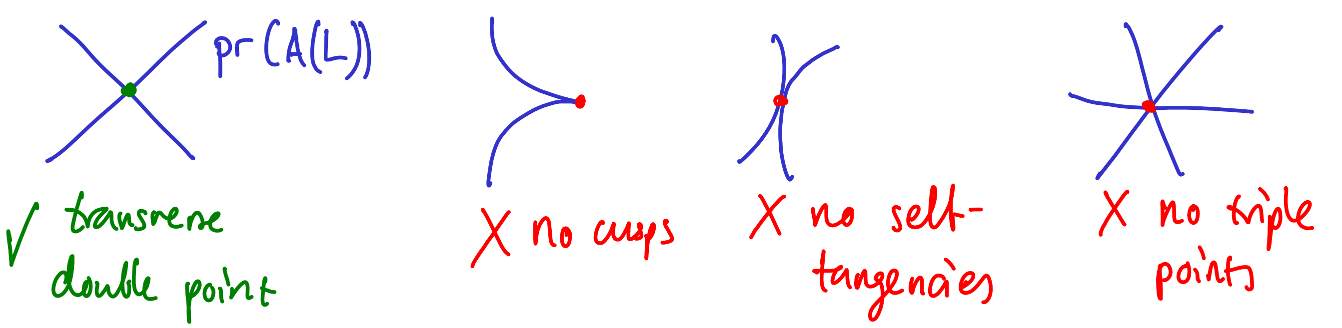
$\square$

2. Knot projections & Knot diagrams

Goal: reduce the isotopy problem to "combinatorics".

Consider the orthogonal projection $pr : \mathbb{R}^{2+k}_{x,y,z,\dots} \longrightarrow \mathbb{R}^2_{x,y}$

Thm. 19 (Jet transversality) For a link $L \subseteq \mathbb{R}^{2+k}$ (one-dim.), any generic $A \in SO(2+k)$ (i.e. $A \in U_L \subseteq SO(2+k)$, U open & dense) has the property that $pr(A(L))$ is an immersion with only transverse double points.

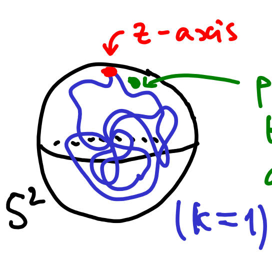


Idea Parametrise $K \subseteq \mathbb{R}^{2+k}$ by an immersion

$$\gamma : S^1 = \mathbb{R}/2\pi\mathbb{Z} \longrightarrow K \subseteq \mathbb{R}^{2+k}$$

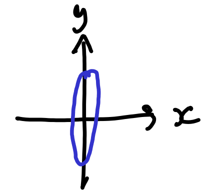
$$\frac{d\gamma}{d\theta} \Big/ \left\| \frac{d\gamma}{d\theta} \right\| : S^1 \longrightarrow S^{k+1}$$ (curve)

has image w. complement that is open & dense by Sard's thm. when $k > 0$



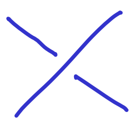
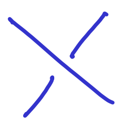
projection along this axis gives an immersion

Ex For $K = \{0\} \times S^1 \subseteq \mathbb{R}^3$, $pr(K) = \{0\} \times [-1,1] \subseteq \mathbb{R}^2$, but a small generic rotation gives



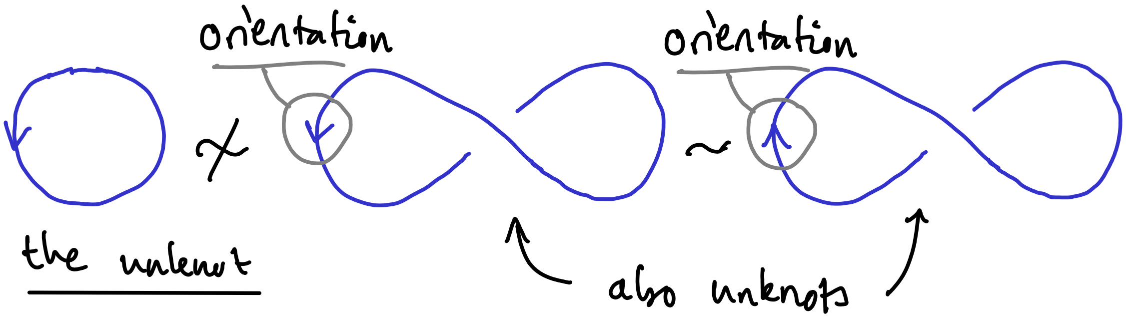
A <u>knot diagram</u> is an immersed curve in $\mathbb{R}^2$ with

- only transverse double points, called <u>crossings</u>, and
- additional data: which branch is above / below at each crossing

e.g.:  or  $\rightsquigarrow$ $\boxed{\text{smooth iso. class of a link.}}$

(with pr = diagram)

For the study of parametrized knots, we also equip each component with an <u>orientation</u>.



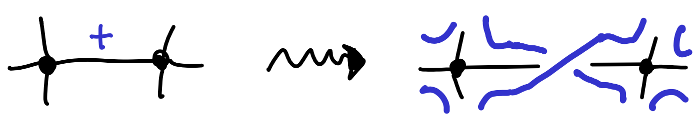
Two knot diagrams are <u>isotopic</u> ($\sim$) if they differ by an ambient isotopy of $\mathbb{R}^2$.

Isotopic knot diagrams clearly give rise to isotopic links in $\mathbb{R}^3$.

The question whether two diagrams in $\mathbb{R}^2$ are isotopic can be reduced to combinatorics. E.g. use polygonal approximations, Riemann mapping thm, isotopy extension thm....

Exercise 18.) Classify all link diagrams without crossings up to isotopy of diagrams.

Exercise 19.) Show that there is a bijective corr. between connected knot diagrams/iso. & embedded conn. graphs $\subseteq \mathbb{R}^2$ w. edges coloured by $+,-$/iso. Which planar directed graphs corr. to knots?

Hint: 

Exercise 20.) Show that there are precisely two diffeomorphisms $\gamma : S^1 \longrightarrow S^1$ up to isotopy. In particular, two parametrisations of a knot

$$\gamma_i : S^1 \longrightarrow K \subseteq \mathbb{R}^n , \quad i = 0,1,$$

which induce the same orientation are always isotopic.

⚠ Some knots are not invertible, i.e. there is no isotopy from K to itself that reverses orientation.