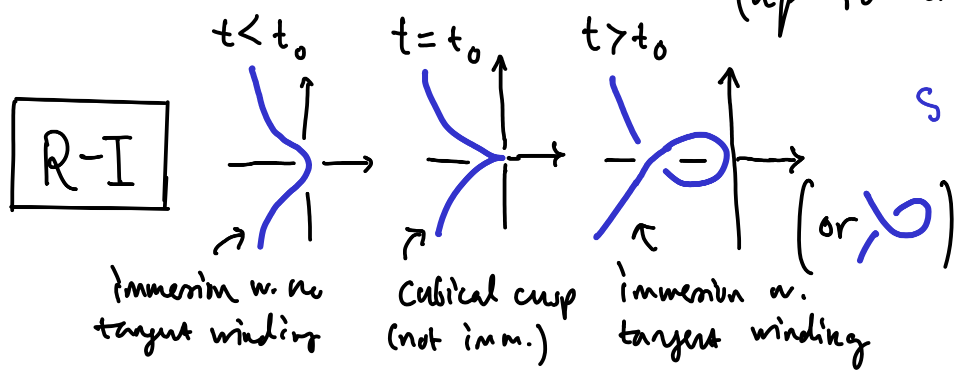
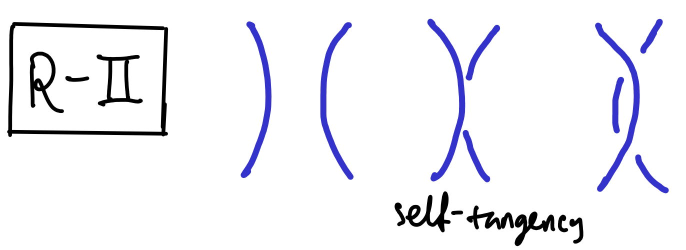
Reidemeister moves

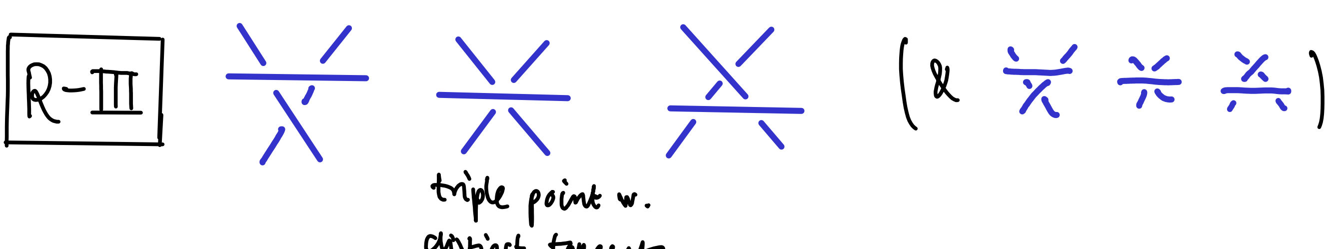
Goal: find a combinatorial model for smooth isotopies.

Thm. 20 (Reidemeister '26) Let $L_t \subseteq \mathbb{R}^{2+k}$ be a smooth isotopy of links, where $pr(L_i) \subseteq \mathbb{R}^2$, $i=0,1$, are immersions w. only transverse double points. After a (generic) C^∞ perturbation of L_t for $t \in (0,1)$, we may assume that $pr(L_t)$ is generated by a smooth isotopy of $\mathbb{R}^2$ except for a finite number of singular moments of the types:
(up to smooth isotopy)



$s \mapsto (s^2, s^3 - ts)$
(cusp at $t=0$)



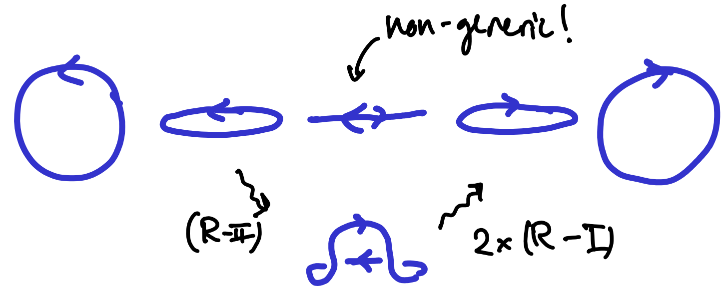


In other words:

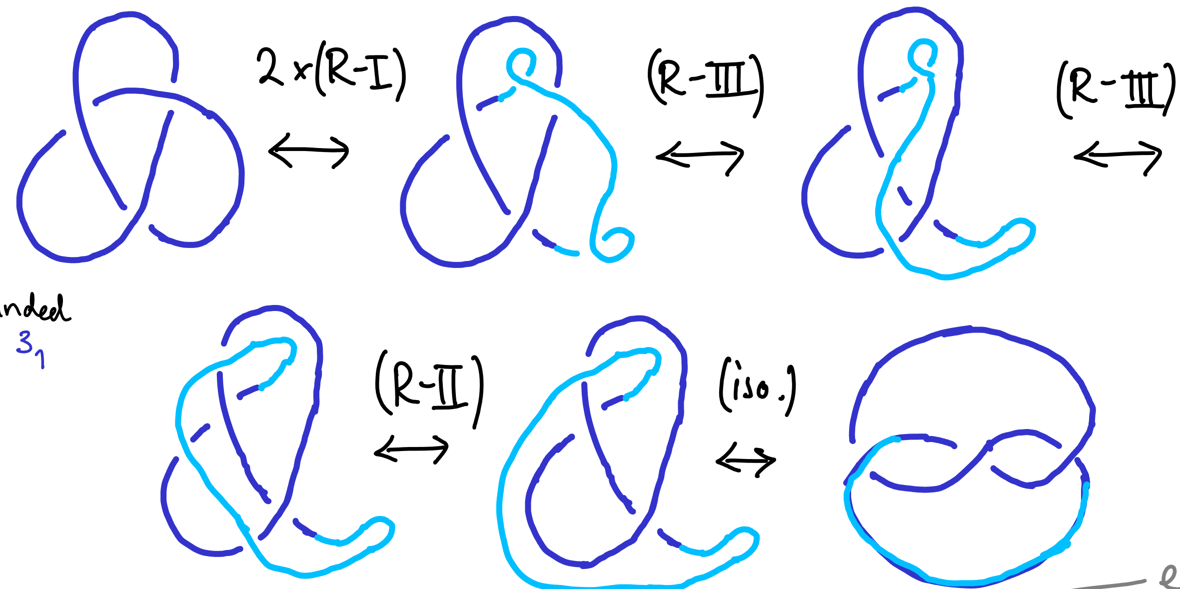
$$\boxed{\begin{array}{l}\text{links} \subseteq \mathbb{R}^3 \,/\, \text{isotopy} \xleftrightarrow{\text{(bij.)}} \\ \quad \text{Knot diagrams} \,/\, \text{isotopy \& (R-I) -- (R-III)}\end{array}}$$

Ex Rotation of planar unknot

perturbation ⇝



Ex



left handed trefoil 3_1

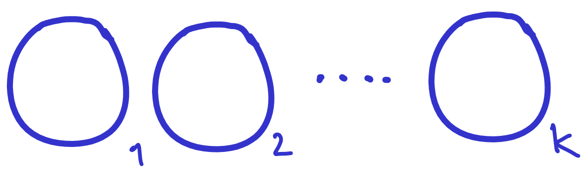
[Coward & Lackenby '11]: An upper bound on the necessary nr. of Reidemeister moves needed to pass from two projections corr. to isotopic knots, given in terms of the nr. of crossings of the two projections.

enormous!

⇒ the isotopy problem is <u>solvable</u> (in theory at least)

Def The <u>crossing number</u> of a link $L \subseteq \mathbb{R}^3$ is the minimal number of crossings for its knot diagrams.

Def The <u>unknotting number</u> of $L \subseteq \mathbb{R}^3$ is the minimal number of "crossing changes"  needed to make it isotopic to the unlink, in addition to (R-I) – (R-III). (a finite nr!)



the unlink of k components.

<u>Exercise 21.)</u> Any two one-dim knots in $\mathbb{R}^n$ are isotopic when $n \geq 4$.

Rem The same is true whenever $M^m \overset{\text{submfd.}}{\hookrightarrow} \mathbb{R}^{2m+1}$, $m \geq 2$.

Colourings & Quandles

A <u>quandle</u> is a pair $(Q, \triangleright)$ for which $\triangleright : Q \times Q \longrightarrow Q$ satisfies:

(Q1) $a \triangleright a = a$

(Q2) $- \triangleright a : Q \to Q$ is <u>bijective</u>

(Q3) $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$ (right self-distributive)

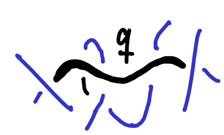
<u>Ex</u> 1.) $Q = G$ group, $\boxed{x \triangleright y := y^{-1} \cdot x \cdot y}$

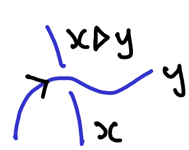
2.) $Q = \mathbb{Z}_p$, $p \geq 3$ prime

$$\boxed{x \triangleright y := 2x - y}$$

(Q3): $2(2a-b)-c = 4a-2b-c$

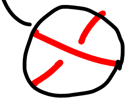
$2(2a-c)-2b+c = 4a-2b-c$

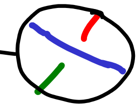
<u>Def.</u> A <u>Q-colouring</u> of an oriented knot diagram is the assignment of an element $q \in Q$ to each arc 

subject to the relation 

at each crossing

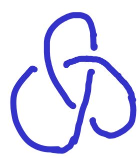
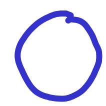
For colourings in $\mathbb{Z}_3$: condition at crossing $\longleftrightarrow$ either all colours same or all three colours used





Also $(x \triangleright y) \triangleright y \equiv 2(2x-y)-y \equiv 4x-3y \equiv x \pmod 3$ $\Rightarrow$ indep of orientation

Thm. 21 The number of Q-colourings is invariant under (R-I) — (R-III) and is therefore a smooth isotopy invariant of links.

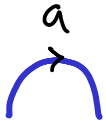
Exercise 22.) 1.) Show that  $\not\approx$  by considering the number of $\mathbb{Z}_3$-colourings (3-colourings)

(in particular: crossing nr (trefoil) $=3$, unknotting nr (trefoil) $=1$)

2.) Classify all knots & links of crossing nr ≤ 3 up to isotopy and mirroring $z \mapsto -z$.

Proof

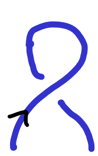
(R-I):



$\rightsquigarrow$

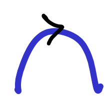
$a \triangleright a = b$
$\underset{(Q1)}{\Rightarrow}$ $b = a$ ✓

(similarly for other orientation)

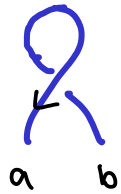


$\Rightarrow$ $b \triangleright a = a$
$(Q1, Q2) \Rightarrow b = a$

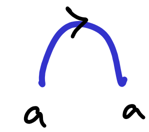
$\rightsquigarrow$



✓



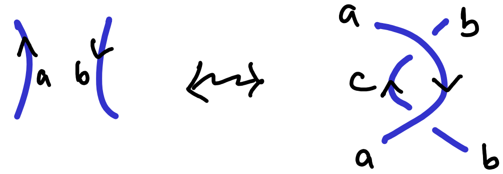
$\Rightarrow b = a \triangleright a = a$ (Q1) $\rightsquigarrow$



✓

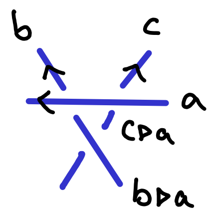
(R-II):

$c = b \triangleright a$
$\quad = b' \triangleright a$
(Q2) $\Rightarrow b = b'$ ✓

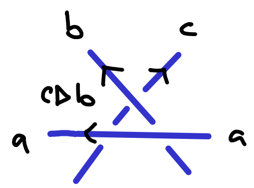


c unique sol. to $c \triangleright a = b$ ✓

(R-III):



$(c \triangleright a) \triangleright (b \triangleright a)$

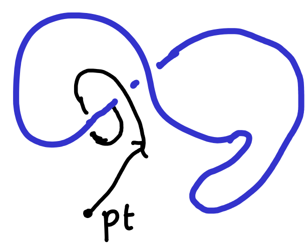


$(c \triangleright b) \triangleright a$
$\|$ (Q3)
$(c \triangleright a) \triangleright (b \triangleright a)$

✓

(etc...) $\square$

<u>Facts</u> (Joyce & Matveev)



- $\exists$ "universal quandle" for each knot
(Q = homotopy classes of paths from a fixed $pt \in \mathbb{R}^3 \setminus L$ that end on the knot)
- This is a complete (but not so useful) knot invariant.