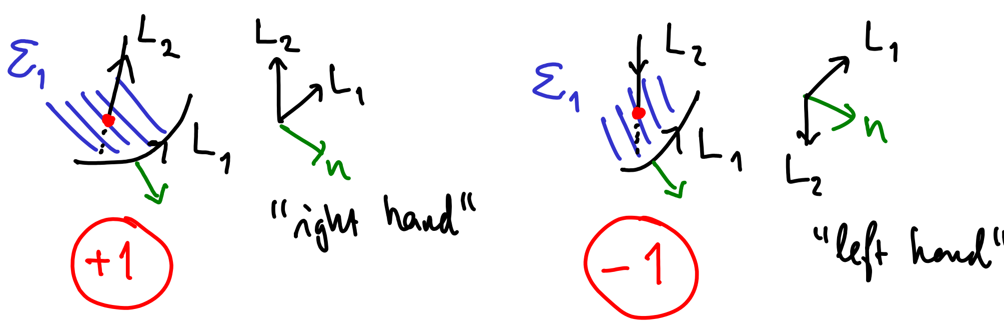
3. Linking number

Given two <u>disjoint</u> <u>oriented</u> links $L_1, L_2 \subseteq \mathbb{R}^3$ we can associate an integer $lk(L_1, L_2) \in \mathbb{Z}$ by the following different constructions. (Useful invariant for Exc. 22.2)

First definition:

(1) Construct an oriented connected compact surface Σ_1 with boundary $\partial \Sigma_1 = L_1$.
(next lecture: construct <u>embedded</u> such surface)

(2) Sum



over all intersections $\Sigma_1 \cap L_2$
for a generic perturbation of L_2 (making L_2 & Σ_1 transverse)

Second definition

Gauß' integral formulation: (c.f. winding nr. in Lecture 3)

$$\boxed{\mathrm{lk}(\gamma_1, \gamma_2) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\gamma_1(\theta) - \gamma_2(\varphi)}{|\gamma_1(\theta) - \gamma_2(\varphi)|^3} \bullet (\dot\gamma_1(\theta) \times \dot\gamma_2(\varphi))\, d\theta\, d\varphi}$$

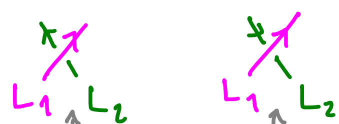
parametrisations of two knots

(Clearly: $\mathrm{lk}(\gamma_1, \gamma_2) = \mathrm{lk}(\gamma_2, \gamma_1)$, c.f. Cor 23 below)

Third definition: via the link diagram

In a link diagram we can define

$$\boxed{\mathrm{lk}(L_1, L_2) = \sum 1 - \sum 1 \in \mathbb{Z}}$$



up to "rotation"

(ignore all other types of crossings)

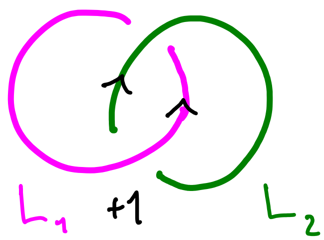
change of orientation of all components

Lma. 22. $\mathrm{lk}((-1)^i L_1, (-1)^j L_2) = (-1)^{i+j}\, \mathrm{lk}(L_1, L_2)$

change of orientation

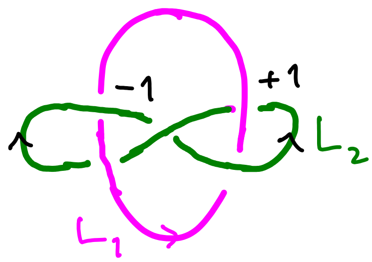
Proof. Immediate.

$\underline{\underline{\text{Ex}}}$



$lk = +1$

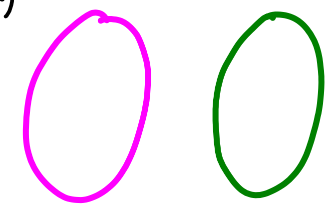
the Hopf link



$lk = 0$

the Whitehead link

(nontrivial) $\not\sim$



the unlink

Exercise 23.) Show that lk is an invariant of smooth isotopy of the link $L_1 \sqcup L_2$, i.e. if there is an isotopy $\varphi_t(L_1 \sqcup L_2)$ s.t. $\varphi_1(L_i) = L_i'$, then $lk(L_1, L_2) = lk(L_1', L_2')$.

Cor. 23. 1.) If $L_1 \sqcup L_2$ is <u>unlinked</u>, i.e $\exists$ isotopy φ_t s.t. $\varphi_1(L_1) \subseteq \{x > 0\}$ & $\varphi_1(L_2) \subseteq \{x < 0\}$, then $lk(L_1, L_2) = 0$

2.) $lk(L_1, L_2) = lk(L_2, L_1)$

<u>Proof</u> 1.) Direct consequence of Exc. 23.

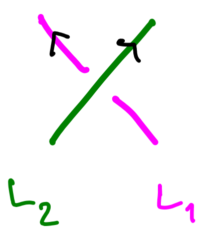
2.) Consider the isotopy L_1^t that translates L_1 by $t \in \mathbb{R}$ in the z-coordinate

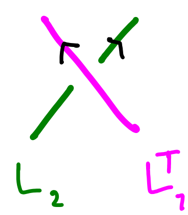
L_1^T, L_2 unlinked when $T \gg 0 \overset{(1.)}{\Rightarrow} lk(L_1^T, L_2) = 0.$

But on the other hand, we compute

$$lk(L_1^T, L_2) = lk(L_1, L_2) - lk(L_2, L_1)$$

Second term comes from the contributions:





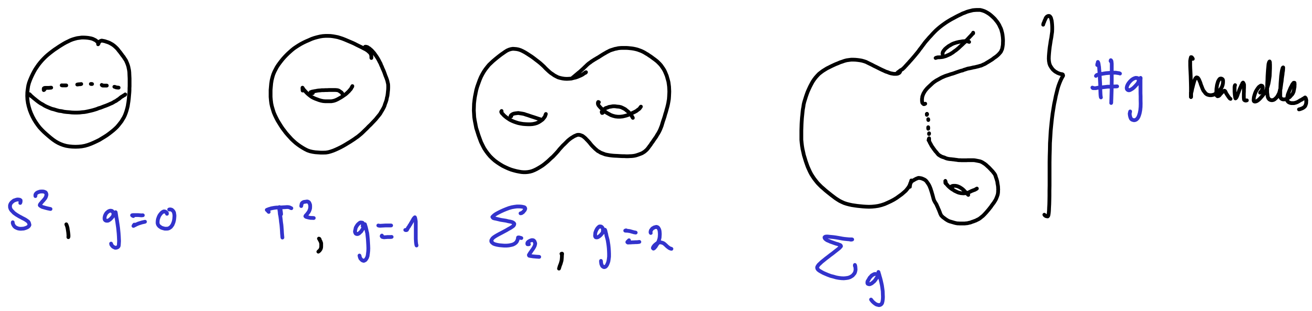
, etc.

(+1) for $lk(L_2, L_1)$ (-1) for $lk(L_1^T, L_2)$

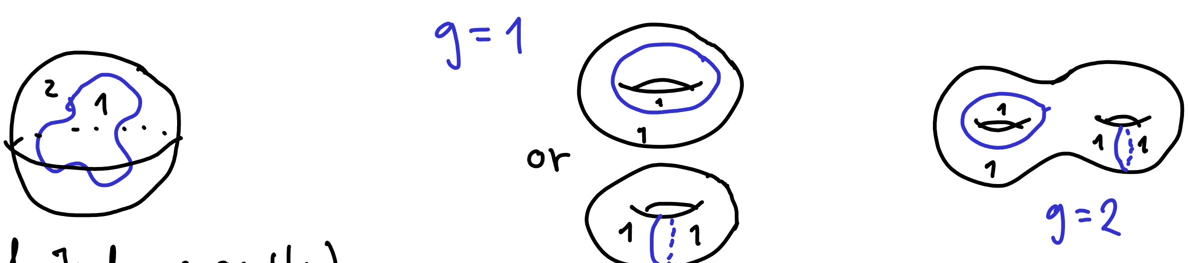
$\square$

3. Seifert surfaces

Recall that the closed oriented surfaces / homeomorphism (or / diffeomorphism) are classified by their <u>genus</u>



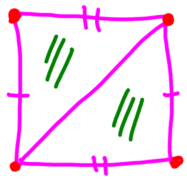
Facts • $g \geq 0$ is the max. nr. s.t. $\underbrace{S^1 \amalg \dots \amalg S^1}_{\# g} \hookrightarrow \Sigma_g$ in a <u>one-sided</u> submfd.



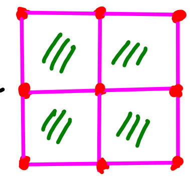
(c.f. Jordan curve thm)

- <u>Euler characteristic:</u> $\chi(\Sigma_g) = \#\text{Vertices} - \#\text{Edges} + \#\text{Faces} = 2-2g$

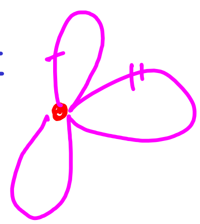
  in <u>any</u> polygonal decomposition (e.g. triangulation), or more generally: any embedding of a finite graph $\Gamma \hookrightarrow \Sigma$ s.t. $\Sigma \setminus \Gamma$ is a disjoint union of <u>open balls</u> up to homeomorphism.

T^2 

$\chi = 1 - 3 + 2 = 0$

T^2 

$\chi = 4 - 8 + 4 = 0$

$\Gamma =$ 

$\Gamma =$ 

The case with boundary

Def. A manifold with boundary is a subset $M \subseteq \mathbb{R}^n$ for which each $pt \in M$ has a nbhd. $\mathcal{U} \subseteq \mathbb{R}^n$ with a diffeomorphism $\Phi : \mathbb{R}^n \xrightarrow{\cong} \mathcal{U}$, $\Phi(0) = pt$, s.t.:

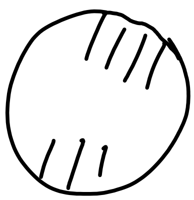
(1) $\Phi^{-1}(M \cap \mathcal{U}) = \mathbb{R}^{\dim M} \times \{0\} \subseteq \mathbb{R}^n$, or

(2) $\Phi^{-1}(M \cap \mathcal{U}) = \left(\mathbb{R}^{\dim M} \cap \{x_1 \geq 0\}\right) \times \{0\} \subseteq \mathbb{R}^n$.

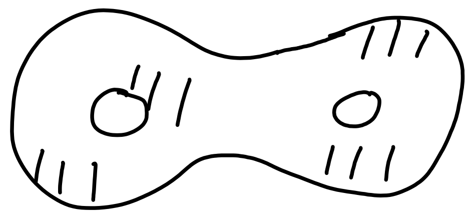
Fact: cases are mutually exclusive!

The boundary $\partial M \subseteq \mathbb{R}^n$ of M is the $(\dim M - 1)$-dimensional submanifold parametrised by $\left(\mathbb{R}^{\dim M} \cap \{x_1 = 0\}\right) \times \{0\}$ in the above coordinates of type (2).

A cpct oriented surface with $k \geq 0$ boundary components is diffeomorphic to the complement of k open balls in some Σ_g. Denote it by $\Sigma_{g,k}$, $\partial \Sigma_{g,k} = \underbrace{S^1 \amalg \ldots \amalg S^1}_{\# k}$



$\Sigma_{0,0}$

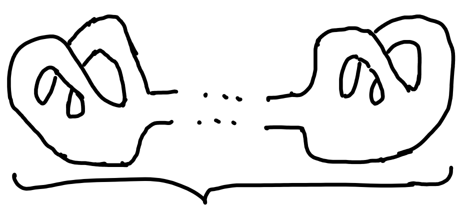


$\Sigma_{0,3}$

<u>planar</u> domains

<u>Exercise 24.)</u> Show that $\Sigma_{1,1} =$

& $\Rightarrow \Sigma_{g,1} =$



g nr

Again, g is the max nr. of components of a non-dividing curve in $\Sigma_{g,k}$

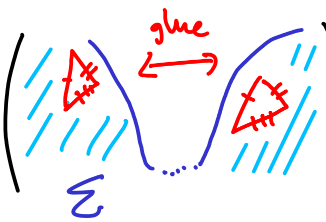
<u>Jordan curve theorem</u>: $\Sigma_{g,k} \subseteq \mathbb{R}^2$ (<u>planar</u> domain)

$\Rightarrow g = 0$.

The computation of χ works as before, the condition $\Sigma \setminus \Gamma \cong B^2 \amalg \ldots \amalg B^2$ (open ball) implies that $\boxed{\partial \Sigma \subseteq \Gamma.}$

Exercise 25.)

1.) $\chi\left(S^2 \setminus \coprod k \text{ balls}\right) = 2-k$

2.) $\chi\Bigg($ 

$\Bigg) = \chi(\Sigma) - 2$

(Obs: Σ can be either connected or disconnected)

3.) $\chi\left(\Sigma_{g,k}\right) = 2-2g-k$