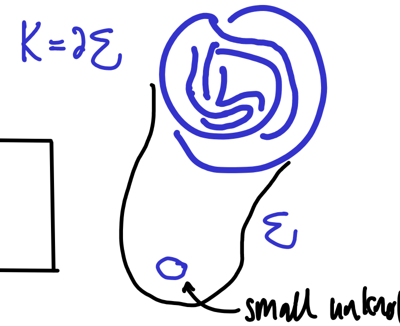
Def A Seifert surface for a link $L \subseteq \mathbb{R}^3$ is a connected oriented surface $\Sigma \subseteq \mathbb{R}^3$ (submfd.) with boundary $\partial\Sigma = L$. The Seifert genus $g(L)$ of L is the minimal genus of any of its Seifert surfaces.



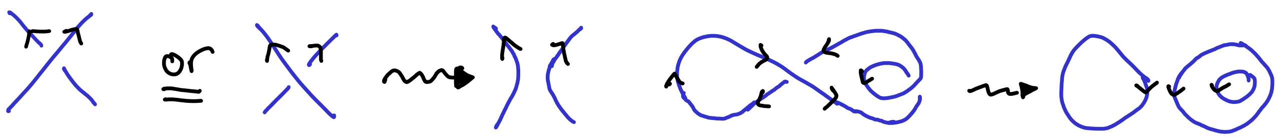
Clearly: $\boxed{K \text{ knot } \; g(K) = 0 \iff K \text{ unknot}}$

We will see that Seifert surfaces exist, so $g(L) \in \mathbb{Z}_{\geq 0}$, and moreover: g is additive under the geometric operation of "connected sum" $\Rightarrow$ infinitely many knots / iso.

How to construct a Seifert surface

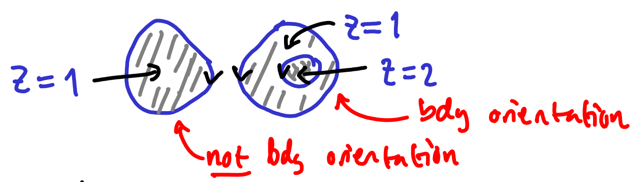
Take a knot diagram for a link $L \subseteq \mathbb{R}^3$. (The surface constructed will depend on this choice) | for knots: choice does not affect Σ.

Step (1): Choose an orientation & resolve crossings by

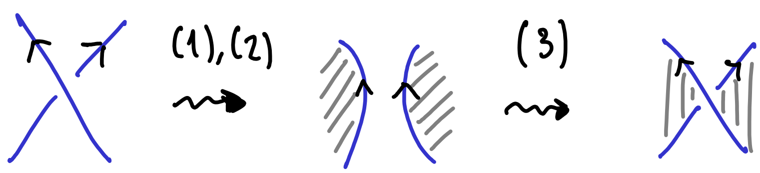


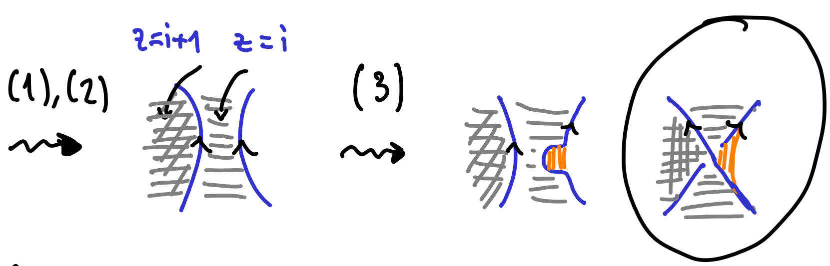
Step (2): We obtain $d > 0$ closed oriented curves that bound d nr. of nested discs $\subseteq \mathbb{R}^2$. (⚠ orientation might differ from bdy orientation)

Lift the disc at the i:th level of the nesting to $\mathbb{R}^3$ by giving it coordinate $z = i$.

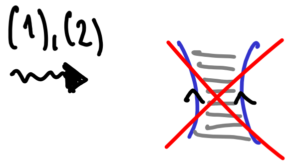


Step (3): Add a twisted band at each crossing according to





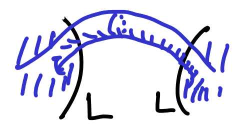
This is a surface Σ with $\partial\Sigma = L$



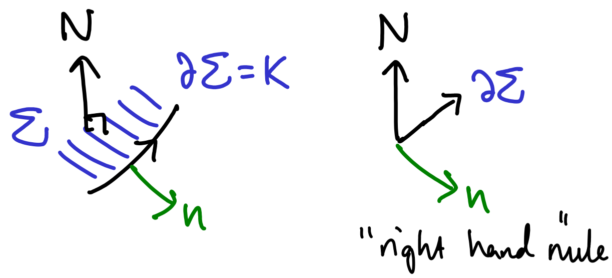
not possible since bdy orientation either agrees or disagrees w. closed curve

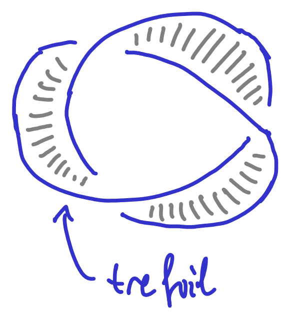
(etc...) $\square$

- The constructed surface Σ is <u>connected</u> when $\partial\Sigma = K$ is connected. <u>Otherwise:</u> connect by "tubes".



- It is <u>orientable</u> since the surface is <u>two-sided</u>, outward normal N can be assigned by

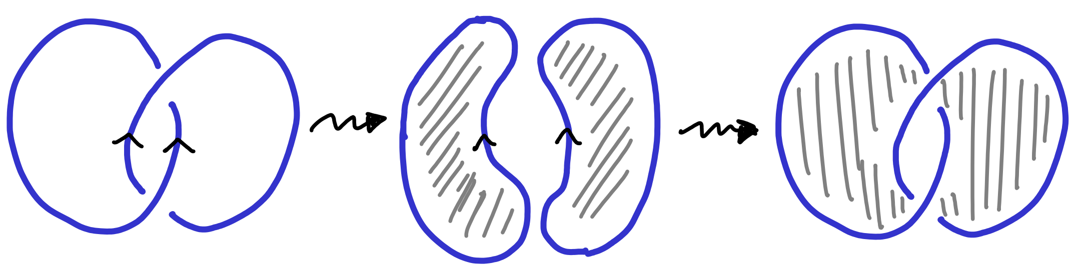




A Möbius band is <u>one-sided</u> and thus not a Seifert surface

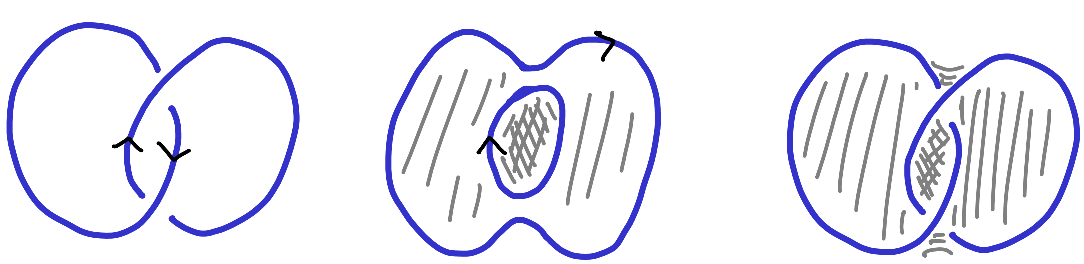
Check that the bands "preserve" this orientation of the nested discs.

<u>Ex</u> Hopf link



(1) (2) (3)

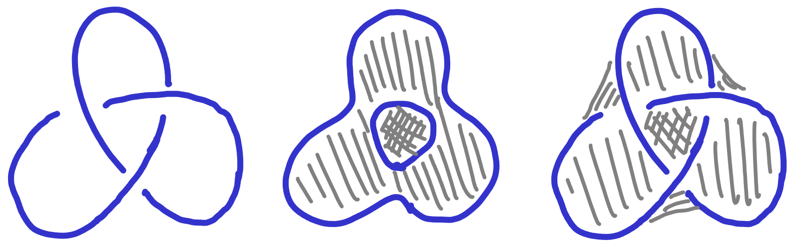
Σ = annulus



(1) (2) (3)

Σ = annulus

Ex Trefoil

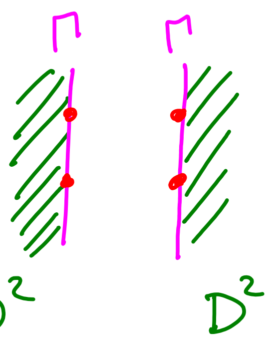


$g=1$ since
- oriented
- conn. bdy

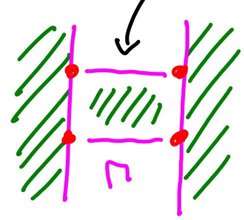
Lem. 24 If the algorithm produces d discs at step (2), and the diagram has q crossings, then $\chi = d - q$.

Proof. $\chi(\underbrace{D^2 \sqcup \cdots \sqcup D^2}_{\#d}) = d.$

w.l.o.g.



band attached in Step (2).

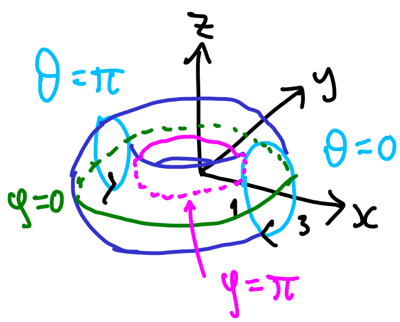


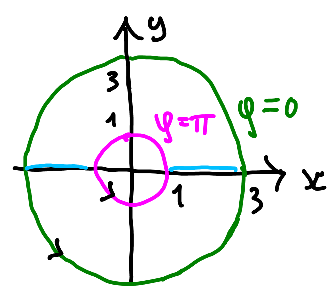
$\chi_{new} = \chi_{old} - 2 + 1 = \chi_{old} - 1$

after attaching bands $\square$

Torus knots & links Consider the "unknotted" torus

$$\mathbb{T}^2 = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} \hookrightarrow \mathbb{R}^3$$
$$(\theta, \varphi) \mapsto ((2+\cos\varphi)\cos\theta, (2+\cos\varphi)\sin\theta, -\sin\varphi)$$





For $p \in \mathbb{Z}_{>0}$, $q \in \mathbb{Z} \setminus \{0\}$, consider the union of the closed curves

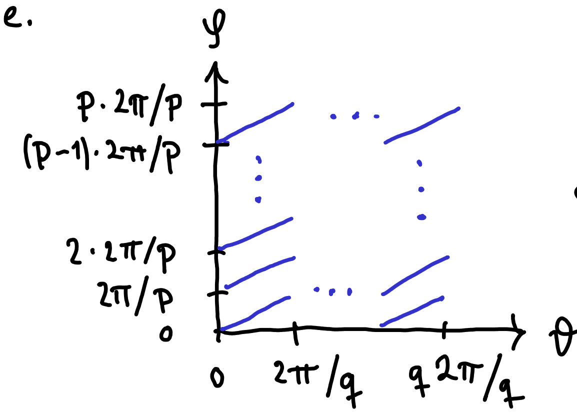
$$t \longmapsto (p\cdot t,\ q\cdot t + m\cdot 2\pi/p) \in \{(\vartheta,\varphi)\} = \mathbb{T}^2$$

$$m = 0, 1, \dots, p-1,$$

$$t \in [0, 1/\gcd(p,q)]$$

⚠ possibly not m distinct curves!

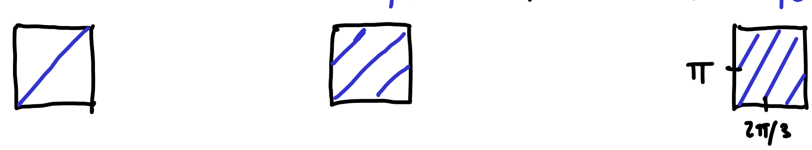
on the torus, i.e.



slope $= q/p$

Its image under the above embedding $\mathbb{T}^2 \hookrightarrow \mathbb{R}^3$ is the <u>(p,q)-torus link</u> $T_{p,q} \subseteq \mathbb{R}^3$.

<u>Ex</u> $T_{1,1}$ = unknot, $T_{2,2}$ = Hopf link, $T_{2,3}$ = right handed trefoil



<u>Exercise 26.</u> Show that $T_{p,q}$ is a link of $\gcd(p,q)$ components.

<u>Exercise 27.</u> Show that the diagram $T_{p,q} \to \mathbb{R}_{x,y}$ gives a Seifert surface of genus $g = (p-1)(q-1)/2$ when $\gcd(p,q) = 1$.

Connected sum

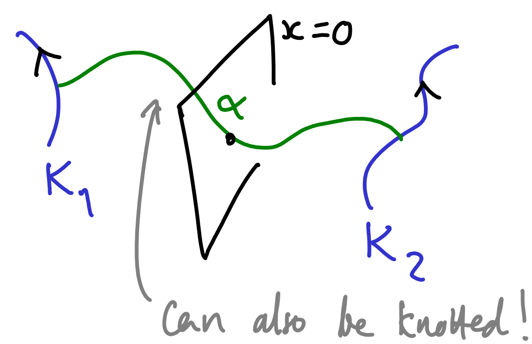
We will show that there are infinitely many knots/iso. by introducing the connected sum operation:

Let $K_1, K_2 \subseteq \mathbb{R}^3$ be oriented knots contained in two disjoint balls,

Step (0): Isotope the balls into the two different half-planes, so that $K_1 \subseteq \{x<0\}$, $K_2 \subseteq \{x>0\}$.

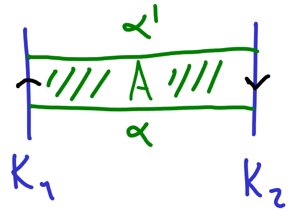
Step (1): Choose an embedded arc $\alpha \subseteq \mathbb{R}^3$ with

- boundary on K_1 & K_2 where its tangent is orthogonal to K_i
- interior disjoint from K_i.
- intersects the hyperplane $x=0$ transversely in a single point



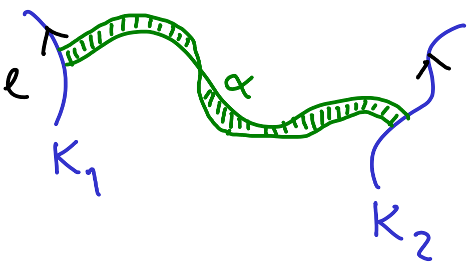
this is crucial for well-definedness!

Step (2): Fatten α to a small band



- One side α, opposite side α' given as a pushoff of α along a nonvanishing v.field.

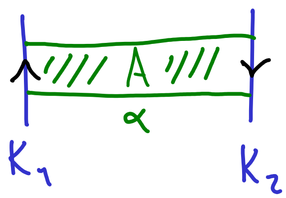
i.e



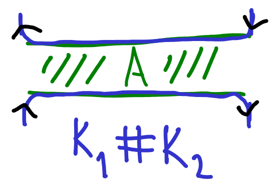
- v.field pos. tangent to K_1 & neg. tangent to K_2
- band intersects K_i along precisely one side

(there is a $\mathbb{Z}$ worth of choices: how many twists around α)

Step (3): Replace $K_1 \amalg K_2$ by the <u>knot</u>



$\rightsquigarrow$ $K_1 \# K_2 :=$



This finishes the construction of the connected sum

In the knot diagram this can be described as: