Digression: Finitely presented groups

$$G = \langle \underbrace{a_1, \dots, a_m}_{\text{generators}} \mid \underbrace{r_1, r_2, r_3, \dots, r_k}_{\text{relations: words in } a_i^{\pm 1}} \rangle$$

is by definition the quotient of the "free group" (no rel^s)

$$\underbrace{\langle a_1, \dots, a_m \rangle}_{\text{free}} / R$$

by R: the smallest normal ($\forall x : xR = Rx$) subgp. that contains all $r_i \in \langle a_1, \dots, a_m \rangle$

The free (nonabelian) group of m generators consists of certain words in the letters $a_1, \dots, a_m$ & $a_1^{-1}, \dots, a_m^{-1}$; as a set it consists of

$$\boxed{\begin{array}{l} 1 : \text{the "empty word"} \\ a_{i_1}^{l_1} \cdot \ldots \cdot a_{i_n}^{l_n} : \text{word of length } l_1 + \ldots + l_n \\ \quad \text{s.t. } l_j \in \mathbb{Z} \setminus \{0\},\ i_j \neq i_{j+1}. \end{array}} \quad (*)$$

$\# |l_1|$ letters $a_{i_1}^{\operatorname{sgn} l_1}$

Multiplication: concatenation of words, followed by the simplification

$$\underbrace{A \cdot a_{i_{n_0}}^{l_{n_0}} \cdot \ldots \cdot a_{i_n}^{l_n}}_{\text{first word}} \cdot \underbrace{a_{j_1}^{l'_1} \cdots a_{j_{n-n_0+1}}^{l'_{n-n_0+1}} \cdot B}_{\text{second word}} \sim A \cdot B \leftarrow \text{word of form } (*)$$

subwords

whenever $n_0 \leq n$ is the smallest index s.t. $i_{n_0+\alpha} = j_{n-n_0+1-\alpha}$, $l_{n_0+\alpha} = -l'_{n-n_0+1-\alpha}$ $\forall \alpha \geq 0$

Later today: geometric realisation

Ex

- $\langle a \rangle \cong \mathbb{Z}$

  i.e. $(a_i \cdot a_j)(a_j \cdot a_i)^{-1}$
  or $a_i \cdot a_j = a_j \cdot a_i$

- $$G = \langle a_1, \dots, a_m \mid \overbrace{a_i \cdot a_j \cdot a_i^{-1} \cdot a_j^{-1}}, \; i,j \in \{1, \dots, m\} \rangle \cong \mathbb{Z}^m$$

  - $\psi : \mathbb{Z}^m \longrightarrow G$, $(l_1, \dots, l_m) \mapsto [a_1^{l_1} \cdot \ldots \cdot a_m^{l_m}]$ is well-def. & <u>surjective</u>
  - $\langle a_1, \dots, a_m \rangle \xrightarrow{\Phi} \mathbb{Z}^m$ has $R \subseteq \ker \Phi$

    $a_i \longmapsto (0, \dots, 0, \underset{(i)}{1}, 0, \dots, 0)$

    $\Rightarrow$ descends to $\varphi : G \to \mathbb{Z}^m$

  Iso. follows from $\psi \circ \varphi = id_G$, $\varphi \circ \psi = id_{\mathbb{Z}^m}$ (easily checked)

- All finite groups are finitely presented

  $$G \cong \langle a_g \;\; \forall\, 1 \neq g \in G \mid (a_g \cdot a_h) \cdot a_{g \cdot h}^{-1} \;\; \forall g \neq 1 \neq h \rangle$$

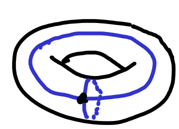
<u>Problem:</u> the word problem is undecidable, so it is algorithmically difficult to work with presentations.

One way to work with groups: realize them as $\pi_1(X)$.

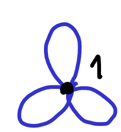
Thm. 27. $\pi_1(\underbrace{S^1 \vee \dots \vee S^1}_{m}) \cong \langle a_1, \dots, a_m \rangle$

$$S^1 \vee \dots \vee S^1 = \{ (z_1, \dots, z_m) \in (S^1)^m \subseteq \mathbb{C}^m \mid z_i \neq 1 \text{ for at most } \underline{\text{one}}\ i \}$$

"bouquet"/"wedge" of circles



$S^1 \vee S^1$



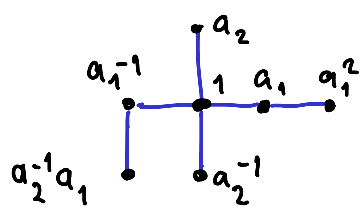
$S^1 \vee S^1 \vee S^1$

<u>Proof</u> Construct a graph Γ with

<u>vertices</u> : elements of $\langle a_1, \dots, a_m \rangle$ (certain words)

<u>edges</u> : one edge between w_1 & w_2 iff $\exists\, i$ s.t. either $w_1 = w_2 \cdot a_i$ or $w_2 = w_1 \cdot a_i$.

<u>Claim 1.)</u> Γ is a <u>tree</u>, and <u>contractible</u> with the standard topology (vertices: discrete topology, then glue edges w. quotient topology)



<u>Claim 2.)</u> The action $\langle a_1, .., a_m \rangle \circlearrowright$ vertices $= \langle a_1, .., a_m \rangle$ extends to a <u>continuous</u> action on Γ s.t.

$$\Gamma / \langle a_1, \dots, a_m \rangle = \underbrace{S^1 \vee \dots \vee S^1}_{m}$$

<u>Exercise 30.)</u> Prove above claims.

LES implies that $\pi_1(S^1 \vee .. \vee S^1) \cong \langle a_1, .., a_m \rangle$. $\square$

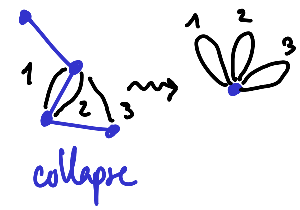
<u>Recall</u>: $\pi_1(S^1) = \mathbb{Z}$, universal cover $\widetilde{S^1} = \mathbb{R}$

we have shown above that $\widetilde{S^1 \vee .. \vee S^1} = \Gamma$ is a tree.

Cor. 28. 1.) $\pi_1(\text{graph})$ is free (possibly infinitely many generators)

2.) Any subgroup of a free group is free (not necc. finitely gen.)

Sketch of proof 1.) Any graph is htpy. eq. to a bouquet of circles, since we can contract a max. embedded subtree



(use Zorn's lemma).

2.)
$$\begin{array}{ccccc}
\pi_1(S^1\vee\ldots\vee S^1) & \hookrightarrow & \overbrace{S^1\vee\ldots\vee S^1}^{\text{univ. cover}} & \twoheadrightarrow & S^1\vee\ldots\vee S^1 \quad (\text{princ. bundle.})\\
(\text{Thm.}27)\ \| & & \|\ (\text{pf. of Thm.}27) & & \|\\
\langle a_1,\ldots,a_m\rangle & & \underset{\text{tree}}{\Gamma} & & \Gamma/\langle a_1,\ldots,a_m\rangle
\end{array}$$

Consider principal bundle $G \hookrightarrow \Gamma \twoheadrightarrow \Gamma/G$

LES $\Rightarrow$ $\pi_1(\Gamma/G) \cong G$

Since Γ/G is a graph, the statement follows by (1.) $\square$

7. The Braid group

The Braid group on n strands Br_n is the group

$$Br_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \overbrace{(\sigma_i \sigma_{i+1} \sigma_i)(\sigma_{i+1}\sigma_i\sigma_{i+1})}^{\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}}{}^{-1}, \quad i = 1, \dots, n-2 \\ \underbrace{(\sigma_i\sigma_j)(\sigma_j\sigma_i)^{-1}}_{\sigma_i\sigma_j = \sigma_j\sigma_i}; \quad |i-j| \geq 2 \end{array} \right\rangle$$

we proceed to find a geometric realisation.

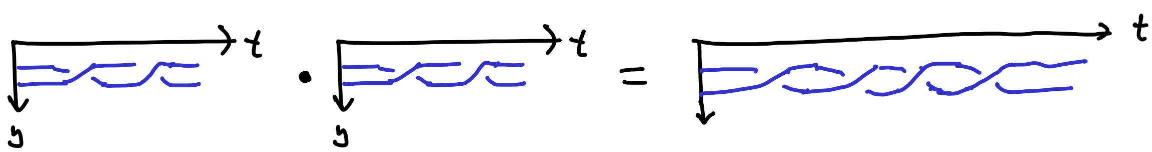
A <u>smooth braid of n strands</u> is a smooth embedding

$$\underbrace{\mathbb{R} \sqcup \dots \sqcup \mathbb{R}}_{n \text{ copies}} \hookrightarrow \underset{\substack{\downarrow \\ t}}{\mathbb{R}} \times \underset{\substack{\Downarrow \\ (x,y)}}{\mathbb{R}^2}$$ which satisfies

- coincides w. $\mathbb{R} \times (\mathbb{R} \times \{1, 2, \dots, n\})$ outside of some cpct subset.
- tangents always have a nonzero t-component (i.e. never tangent to $\mathbb{R}^2$)

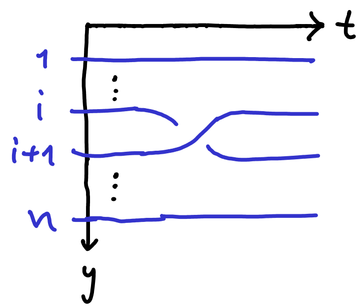
By a <u>smooth isotopy of braids</u>, we mean a compactly supported isotopy <u>through braids</u> (no $\mathbb{R}^2$ tangencies).

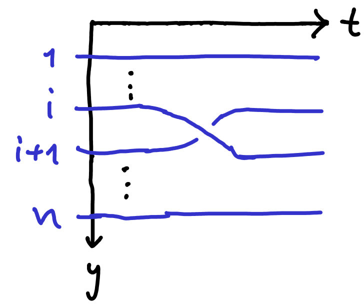
Concatenation $[0,1] \times \mathbb{R}^2 \cup [1,2] \times \mathbb{R}^2$ makes this into a <u>group</u> w. unit $= \mathbb{R} \times \mathbb{R} \times \{1, 2, \dots, n\} \subseteq \mathbb{R} \times \mathbb{R}^2$.



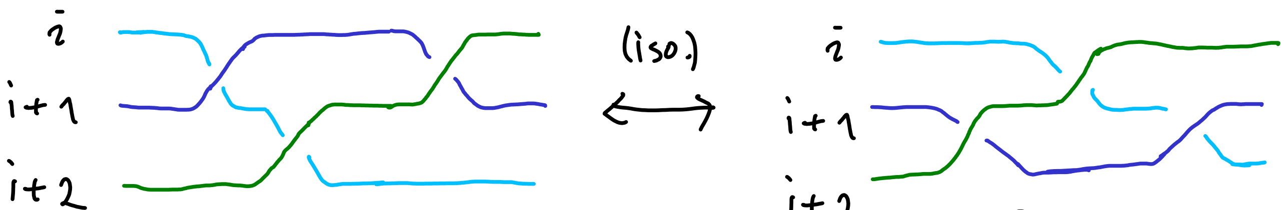
Thm. 29. $Br_n \xrightarrow{\cong}$ Smooth n-Braids / smooth iso. through braids

is an isomorphism of groups.

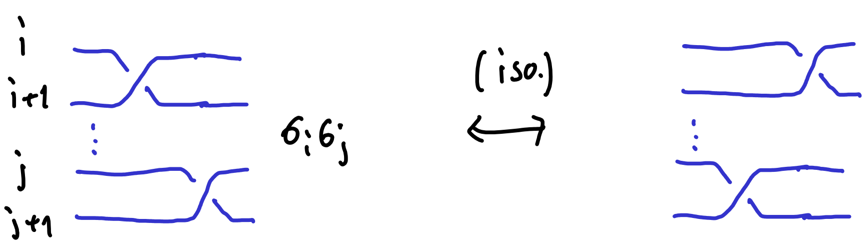
$\sigma_i \mapsto$ 

$\sigma_i^{-1} \mapsto$ 

Idea: • For well-definedness, check relations, e.g.

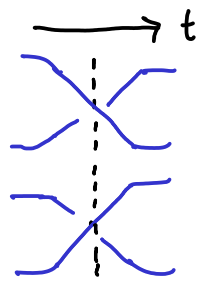


$\sigma_i \sigma_{i+1} \sigma_i$ $\qquad$ $\sigma_{i+1} \sigma_i \sigma_{i+1}$

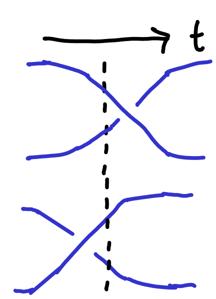
$|i-j| \geq 2$ 

$\sigma_i \sigma_j$ $\qquad$ $\sigma_j \sigma_i$

• Use knot projection to $\underset{t}{\mathbb{R}} \times \underset{y}{\mathbb{R}}$ after a generic perturbation to show <u>surjectivity</u>.

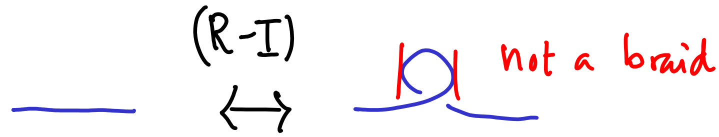


not generic
(two X w. same t-coord)



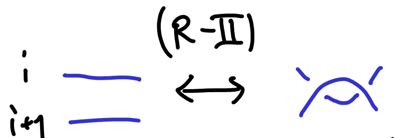
generic

- For <u>injectivity</u> show that (R-II), (R-III), & isotopies of diagrams e.g. can be generated by the relations of Br_n.

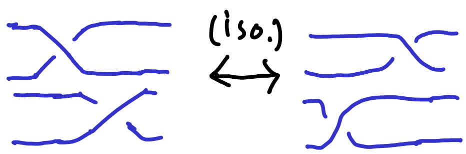


<u>OBS</u>: (R-I) does not appear in an <u>iso. of braids</u>.

<u>E.g.</u>



$\sigma_i \sigma_i^{-1}$

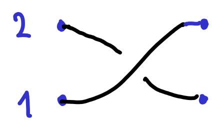


$\sigma_i^{-1} \sigma_{i+1} = \sigma_{i+1} \sigma_i^{-1}$

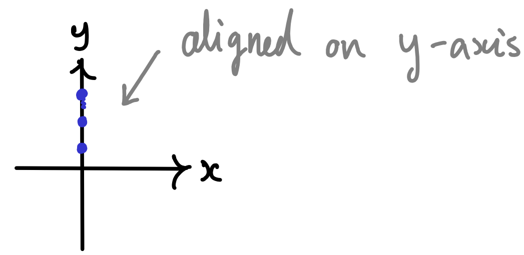
<u>Exercise (31.)</u> Find the relation in Br_n for (R-III)

Cor. 30. $\pi_1(\mathrm{Conf}_n(\mathbb{R}^2)) \cong \mathrm{Br}_n$

Sketch of pf. $\mathrm{Br}_n \to \pi_1(\mathrm{Conf}_n(\mathbb{R}^2))$ is straight-forward in view of Thm. 29; any smooth braid gives rise to a path in $\mathrm{Emb}(\bar{n}, \mathbb{R}^2)$ which becomes a closed curve in $\mathrm{Conf}_n(\mathbb{R}^2)$.



Here the basepoint is chosen as



To construct the inverse, $\pi_1 \to \mathrm{Br}_n$ consider $S^1 \to \mathrm{Conf}_n(\mathbb{R}^2)$

homotopy lifting $\Rightarrow$

$$\begin{tikzcd} & \mathrm{Emb}(\bar{n}, \mathbb{R}^2) \arrow[d] \\ {[0, 2\pi) \subseteq S^1} \arrow[r] \arrow[ur, dotted, "\exists \text{ lift } \varphi"] & \mathrm{Conf}_n(\mathbb{R}^2) \end{tikzcd}$$

φ is a cont. family of n distinct ordered points in $\mathbb{R}^2$

A smooth approximation $\tilde{\varphi}$ of φ gives the braid that consists of the union of traces

$$(t, \mathrm{pr}_{\mathbb{R}^2} \circ \tilde{\varphi}(i)) \in \mathbb{R} \times \mathbb{R}^2$$

$\square$