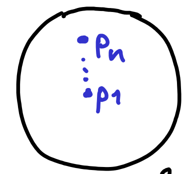
Rmk The isotopy extension thm. shows that

$$\mathrm{Diff}^{\partial}(D^2) \xrightarrow{\pi} \mathrm{Conf}_n(D^2) \sim \mathrm{Conf}_n(\mathbb{R}^n)$$



D^2

$$\pi(\varphi) = \varphi(p_1) \cup \varphi(p_2) \cup \ldots \cup \varphi(p_n)$$

gp. of (oriented) diffeomorphisms $\varphi : D^2 \xrightarrow{\cong} D^2$
(topology: uniform C^1-topology) s.t. $\varphi|_{\partial D^2} = id_{\partial D^2}$

is a "principal bundle" with fibre given by the subgroup

$$\mathrm{Diff}^{\partial}(D^2 \text{ rel. } \{p_1, \ldots, p_n\}) \subseteq \mathrm{Diff}^{\partial}(D^2)$$

i.e. the <u>stabiliser</u> of the action.

diffeo^s fixing $\{p_1, \ldots, p_n\}$ setwise & ∂D^2 pointwise

Since $\mathrm{Diff}^{\partial}(D^2)$ satisfies $\pi_i = 0$ $\forall i$ [Smale],
($\underline{\pi_0 = 0:}$ "all coordinate systems of D^2 are isotopic".)

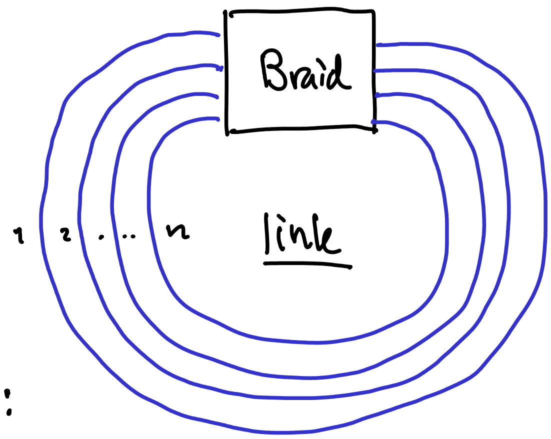
the LES now implies that

$$\boxed{\pi_0\left(\mathrm{Diff}^{\partial}(D^2 \text{ rel } \{p_1, \ldots, p_n\})\right) \cong \pi_1\left(\mathrm{Conf}_n(\mathbb{R}^2)\right) \cong Br_n}$$

this is the same as the quotient

$$\mathrm{Diff}^{\partial}(D^2 \text{ rel } \{p_1, \ldots, p_n\}) \Big/ \underbrace{\mathrm{Diff}^{\partial,0}(D^2 \text{ rel } \{p_1, \ldots, p_n\})}_{\text{isotopies that fix } \partial \text{ pointwise and } \{p_1, \ldots, p_n\} \text{ setwise}}$$

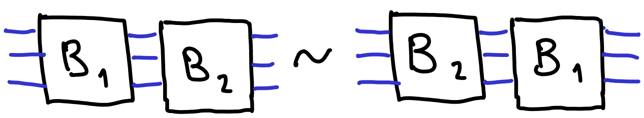
Relations to knots



Markov's thm: $\{Br_n\}_{n=1}^{\infty} \xrightarrow{\text{closure}} \{\text{links}/\text{iso.}\}$

is a bijection if one takes the quotient by the eq. rel. generated by:

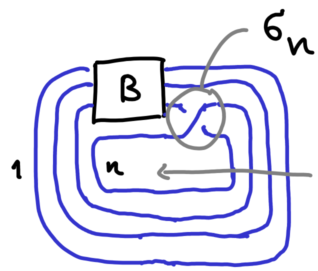
- conjugation in Br_n



$$B_1 B_2 \sim B_2 (B_1 B_2) B_2^{-1} = B_2 B_1$$

- stabilisation

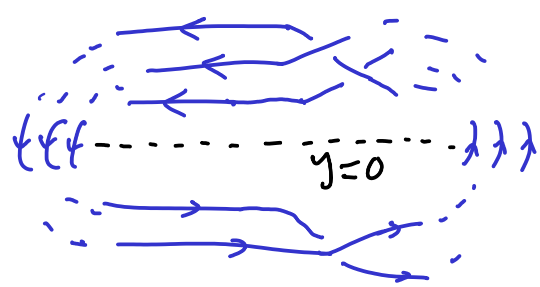
$$\underset{\substack{\cap\\ Br_n}}{B} \sim \underset{\substack{\cap\\ Br_{n+1}}}{B \cdot \sigma_n^{\pm 1}}$$



loop can be removed with (R-I)

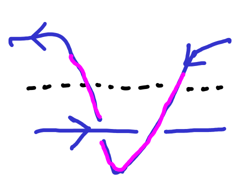
Idea for surjectivity

Goal is to arrange diagram to the form:

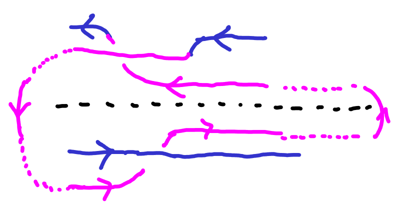


- Pull vertical tangencies to left/right.
- Pull horiz. arcs to correct side of $y=0$ according to their orientation

Obstruction



solution $\rightsquigarrow$



8. Kauffman's version of the Jones polynomial

The <u>Kauffman bracket</u> $\langle \text{link diagram} \rangle \in \mathbb{Z}[A, A^{\pm 1}]$

(i) $\langle \bigcirc \rangle = 1$

(ii) $\langle \bigcirc \sqcup \underset{\text{link}}{L} \rangle = (-A^2 - A^{-2}) \cdot \langle \underset{\text{link}}{L} \rangle$

(iii) $\langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle)(\rangle$ ("skein relation")

The <u>Jones polynomial</u> (Kauffman's version) is

$$\boxed{f_L(A) := (-A)^{-3\underbrace{\left(\sum_{\times} 1 - \sum_{\times} 1\right)}_{\text{the "writhe"}}} \langle L \rangle \in \mathbb{Z}[A, A^{-1}]}$$

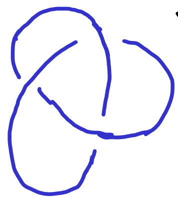
which is an invariant of <u>oriented links / isotopy</u>.

(Unknown if it detects the unknot.)

It has a description as the "trace" of a representation of Br_n built using the Temperley-Lieb algebra.

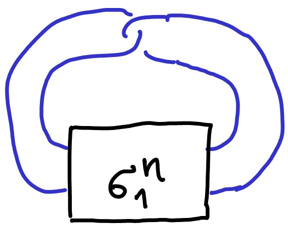
Exercise (32.)

1.) Calculate

$\Big\langle$  $\Big\rangle = -A^{-5} - A^{-3} + A^{-7}$

and the Jones polynomial for <u>both</u> orientations.

2.) Calculate the Jones polynomials of both orientations of



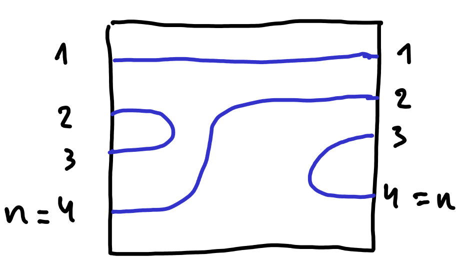
"twist knots" $(n+2)_1$

(<u>$n=1$:</u> trefoil   <u>$n=2$:</u> figure-8 knot

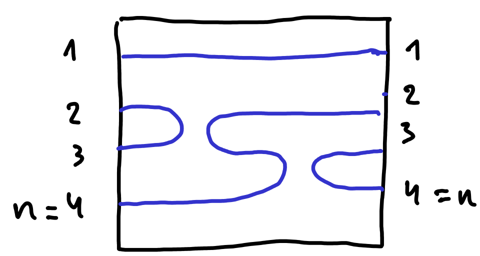
4_1)

Kauffman's realisation of the Temperly-Lieb algebra

For $\delta \in \mathbb{C}$ let $TL^{Kau}(n, \delta)$ be the $\mathbb{C}$-algebra structure on the $\mathbb{C}$-vector space with basis all "crossingless matchings" of n points on $\mathbb{R}$ / isotopy.



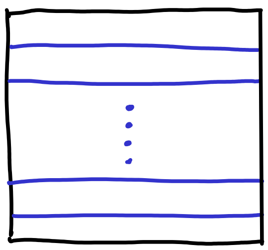
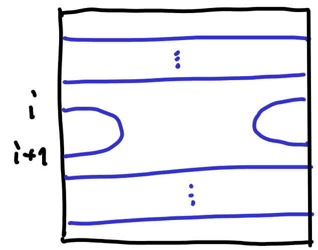
$\sim$



Facts 1.) $\dim TL^{Kau}_n(n, \delta) = \frac{1}{n+1}\binom{2n}{n}$

2.) concatenation + replacing the closed components w. a mult. by scaler $\delta^{\#\text{ closed comp.}}$

yields an associative, unital, algebra structure

$1 =$  $E_i =$ 

3.) The algebra is generated by E_i subject to rel^s:

$E_i^2 = \delta E_i$, $E_i E_{i\pm 1} E_i = E_i$, $E_i E_j = E_j E_i$ if $|i-j| \geq 2$

Parametrise algebras by setting $\boxed{\delta = -A^2 - A^{-2}}$

$$\boxed{\rho(\sigma_i) := A \cdot E_i + A^{-1} \cdot 1}$$

then defines a fam. of representations

$$\rho : Br_n \xrightarrow[\text{gp. morph.}]{} TL^{Kau}(n, -A^2 - A^{-2}) \circlearrowright \mathbb{C}^N \cong TL^{Kau}$$

parametrised by the variable A.

9. More on fundamental groups

Unlike higher homotopy groups, one can typically find a <u>presentation</u> of the fundamental group $\pi_1(X)$.

possible to remove with groupoid formalism

Thm 31. (Seifert van Kampen) If $X = U \cup V$, where $U, V, \underset{\ni pt}{U \cap V} \subseteq X$ <u>open</u> & <u>path connected</u>, then $\exists$ pushout diagram induced by the canonical inclusions $\iota_{\dots}$.

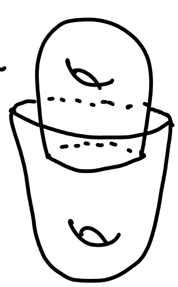
$$\begin{tikzcd}
\pi_1(U \cap V, pt) \arrow[r, "(\iota_{U\cap V, U})_*"] \arrow[d, "(\iota_{U \cap V, V})_*"'] \arrow[dr, phantom, "\circlearrowright"] & \pi_1(U, pt) \arrow[d, "(\iota_U)_*"] \\
\pi_1(V, pt) \arrow[r, "(\iota_V)_*"'] & \boxed{\pi_1(X, pt)}
\end{tikzcd}$$

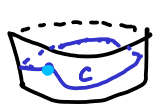
algebraically constructed: $\cong$ amalgamated prod. $\pi_1(U,pt) \underset{\pi_1(U\cap V, pt)}{*} \pi_1(V,pt)$

Cor. 32. If
$$\pi_1(U, pt) = \langle a_1, \dots, a_m \mid r_1, \dots, r_n \rangle$$
$$\pi_1(V, pt) = \langle b_1, \dots, b_k \mid s_1, \dots, s_\ell \rangle$$
$$\pi_1(U \cap V, pt) = \langle c_1, \dots, c_p \mid \dots \rangle$$
then $\pi_1(X, pt) = \langle a_i, b_j \mid r_k, s_\ell, \iota_{U\cap V, U}(c_m) \cdot \iota_{U \cap V, V}(c_m)^{-1} \rangle$

U

V





$U \cap V$

$\iota_{U\cap V, U}(c) = \iota_{U \cap V, V}(c)$ holds in $\pi_1(X)$.

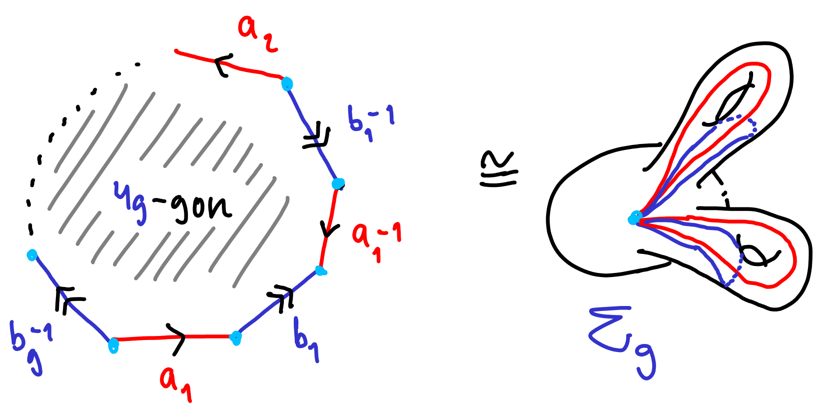
Fundamental group of surfaces

Exercise (33.) Show that:

1.) $\pi_1(\Sigma_{g,k}) \cong \langle a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_{k-1} \rangle, \quad k \geq 1$ (free group)

2.) $\pi_1(\Sigma_g) \cong \langle a_1, \dots, a_g, b_1, \dots, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdot \ldots \cdot a_g b_g a_g^{-1} b_g^{-1} \rangle$

Hint:



Consequence $H_1(\Sigma_g) = \pi_1(\Sigma_g) / [\pi_1(\Sigma_g), \pi_1(\Sigma_g)] \overset{(2.)}{=}$

$$= \langle a_1, \dots, a_g, b_1, \dots, b_g \rangle / [\langle a_i, b_i \rangle, \langle a_i, b_i \rangle]$$

$$= \mathbb{Z}^{2g}$$

The knot group

$\pi_1(\mathbb{R}^3 \setminus L)$ is a powerful knot invariant; it detects the unknot.
("Dehn's lemma")

Exercise (34.) Use smooth approximations and/or the Seifert van Kampen thm. to show that

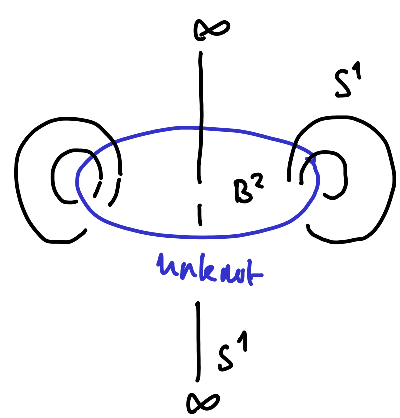
$$\pi_1((\underbrace{S^3 \setminus \{(0,0,1)\}}_{\mathbb{R}^3}) \setminus L) = \pi_1(S^3 \setminus L).$$

("worm holes")
connected sum: glue tubes $I \times S^2$

Since $S^3 \setminus (\text{k-component unlink}) \cong S^1 \times B^2 \# \dots \# S^1 \times B^2$
$\bigcirc \dots \bigcirc$

we conclude that $\pi_1(\mathbb{R}^3 \setminus (\bigcirc \dots \bigcirc)) \cong \langle a_1, \dots, a_k \rangle$

Pf. $S^3 \setminus \underbrace{\{(\cos\theta, \sin\theta, 0)\}}_{\text{unknot}} \cong S^1 \times B^2$

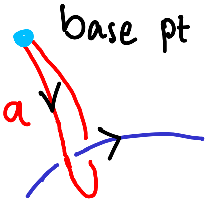


The Wirtinger presentation

$\pi_1(S^3 \setminus L)$ is finitely presented with

generators one for each arc  in a diagram

fix an orientation of L and take

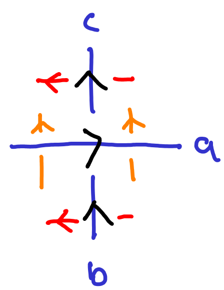


base pt $\in \{z \gg 0\}$

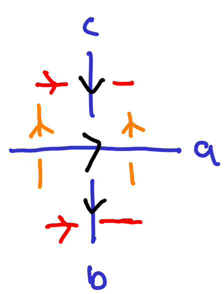
generator a $\in \pi_1$

arc $a \subseteq L$

relations



$c = a^{-1} b a$



$c = a^{-1} b a$

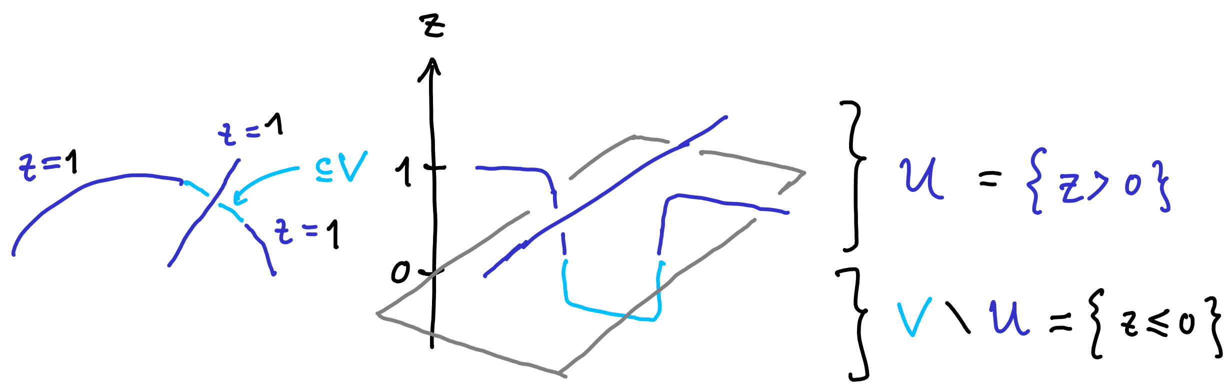
$\Rightarrow$ $\boxed{H_1(S^3 \setminus \text{Knot}) \cong \mathbb{Z}}$

<u>Proof</u> Use Seifert van Kampen with the decomposition

$$S^3 = \underbrace{\{z > 0\}}_{U} \cup \underbrace{\{z < \varepsilon\}}_{V}$$

Where the knot satisfies:

- the arcs in the diagram are contained $\subseteq U$ exept the part of the arc that passes <u>below</u> in the crossing, which in contained in $\subseteq V \setminus U$
- $U \cap V$ intersects each arc in precisely two components (contained near the crossings)



W.l.o.g, assume no <u>closed</u> arc (this is an unlinked unknot that can be moved to the side and handled separately)

$\pi_1(U\cap V,\infty)$ = free gp. gen. by "endpoints of arcs"
(2 × # crossings nr. of gen^{s})

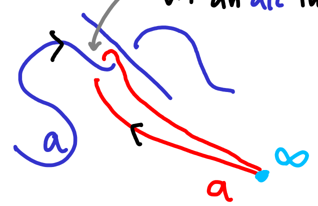
$\pi_1(U,\infty)$ = free gp. gen by arcs

$\pi_1(V,\infty)$ = free gp. gen by "crossings"

point far away

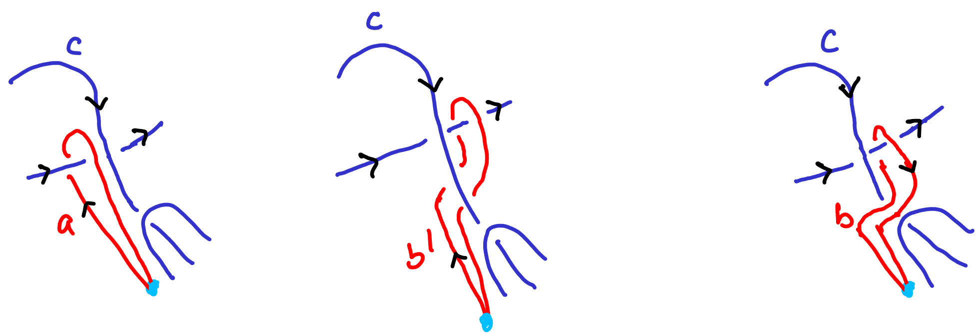
<u>Arc generators</u> all come from $\pi_1(U\cap V,\infty)$
$\cong$
free gp. of 2×crossing
nr. of crossings.

only intersection
w. an arc in proj.



i.e. $\pi_1(U\cap V) \twoheadrightarrow \pi_1(U)$ is surjective

same for $\pi_1(U\cap V) \twoheadrightarrow \pi_1(V)$

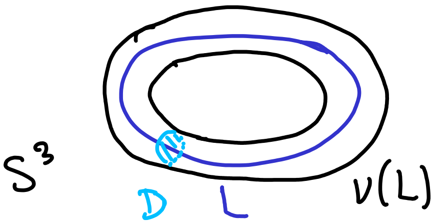


$a = b' \in \pi_1(V)$

$b' = c\, b\, c^{-1} \in \pi_1(U)$

Relations to 3-manifolds

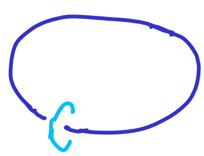
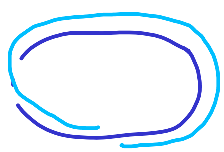
Lickorish-Wallace Theorem Any closed & orientable 3-dim^{l} manifold M^3 can be obtained by surgery on Link $\subseteq S^3$, i.e.

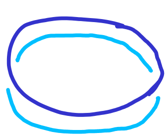


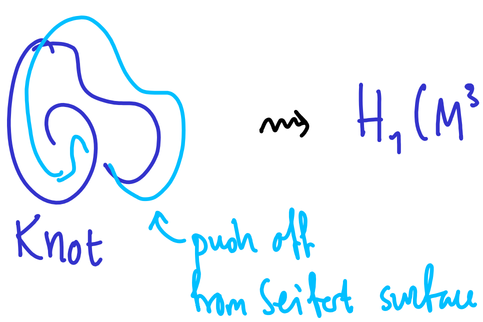
Remove tube $\nu(L)$

Reattach by gluing w. automorphism of $\partial \nu(L)$. $\mathrm{Aut}(\mathbb{T}^2) \sim Gl_2(\mathbb{Z})$

In effect: suffices to prescribe where $\partial D \subseteq \partial \nu(L)$ ends up

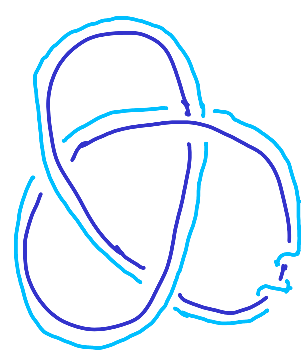
 or  $\rightsquigarrow$ $M^3 \cong S^3$

 $\rightsquigarrow$ $M^3 = S^1 \times S^2$



$\rightsquigarrow$ $H_1(M^3) = 0$,

<u>ex</u>



$\rightsquigarrow$ Poincaré homology sphere