V Connections & Curvature

1. Vector fields

$$T\mathbb{R}^n = \underset{\text{point}}{\mathbb{R}^n} \times \underset{\substack{\text{direction at point}\\ \text{(vector)}}}{\mathbb{R}^n}$$

$pt \bullet \nearrow$ direction X_{pt}

point

$$T_{pt}\mathbb{R}^n := \{pt\} \times \mathbb{R}^n$$

$f : \mathbb{R}^n \to \mathbb{R}^m$ smooth function

$Tf : T\mathbb{R}^n \longrightarrow T\mathbb{R}^m$ smooth function for which

(here: the <u>Jacobi matrix</u>)

$T_{pt} f = Tf\big|_{T_{pt}\mathbb{R}^n} : T_{pt}\mathbb{R}^n \to T_{f(pt)}\mathbb{R}^m$ is the <u>differential</u> of f at pt,

i.e. the linear function $D_{pt} f = \left[\dfrac{\partial f^i}{\partial x_j}(pt) \right]_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \in Mat_{n \times m}(\mathbb{R})$

A vector field is a smooth function
$$\mathbb{R}^n \to T\mathbb{R}^n \quad \in \Gamma(T\mathbb{R}^n)$$
$$pt \mapsto X_{pt} \in \{pt\} \times \mathbb{R}^n$$

In <u>this case</u> we have a canonical identification

functions $\tilde{X} : \mathbb{R}^n \to \mathbb{R}^n \quad \longleftrightarrow \quad pt \mapsto (pt, \tilde{X}_{pt}) = X_{pt}$ vector field

(as in calculus)

$\tilde{X} \equiv (0, \dots, 0, \underset{i}{1}, 0, \dots, 0) \longleftrightarrow$ coordinate vector field "$\dfrac{\partial}{\partial x_i}$"

$M \subseteq \mathbb{R}^N$ submanifold

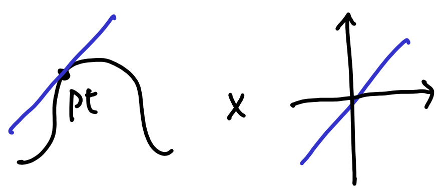
$$TM \overset{\text{def.}}{=} \{ X_p \in \mathbb{R}^N \mid X_p \text{ in image of } T\varphi,\ \varphi \text{ some loc. param} \} \quad (\Rightarrow p \in M)$$

$$\overset{\text{alt. def.}}{=} \{ X_p \in \mathbb{R}^N \mid f|_M \text{ const.} \Rightarrow Tf(X_p) = 0 \} \subseteq M \times \mathbb{R}^N$$

$$T_{pt} M \overset{\text{def.}}{=} TM \cap T_{pt}\mathbb{R}^N \cong \mathbb{R}^{\dim M}$$

not canonical!!!

(there is no canonical parametrisation in general)



$M^1 \subseteq \mathbb{R}^2$ $\qquad$ $T_{pt} M \subseteq \mathbb{R}^2$

Facts:

- $TM \subseteq \mathbb{R}^N \times \mathbb{R}^N$ smooth mfd. w. local param. $T\varphi$ (param of M: φ)
- $f : M \longrightarrow N$ smooth $\rightsquigarrow$ $Tf : TM \longrightarrow TN$ smooth & fibrewise linear
- Chain rule: $T(f \circ g) = Tf \circ Tg$

  Consequence: $T\, id_M = id_{TM}$, and hence

  $f : M \to N$ diffeomorphism

  $\Rightarrow Tf : M \to N$ diffeomorphism

- TM is a (smooth) fam. of vector spaces param. by M.

  A smooth fam. of vectors is a <u>vector field</u>

$$X : M \longrightarrow TM \quad \text{smooth}$$
$$pt \longmapsto X_{pt} \in T_{pt} M$$

We write $\Gamma(TM)$ for the (infinite dimensional) $\mathbb{R}$-vector space of tangent vector fields

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- No canonical Jacobi matrix associated to TM
- Not possible to identify v. fields $M \to TM$ & functions $M \to \mathbb{R}^{\dim M}$ in general.

<u>Obs</u> We can always <u>choose</u> local coords

$$\varphi : \mathbb{R}^n \hookrightarrow N$$
$$\psi : \mathbb{R}^m \hookrightarrow M$$

to reduce computations to local coordinates, e.g.

$$\varphi^{-1} \circ f \circ \psi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

Smooth vector fields $\longleftrightarrow$ One parameter subgroups

$X \in \Gamma(TM)$

$X_p \in T_pM$, $p \in M$

$\{\varphi_X^t\}_t \subseteq \mathrm{Diff}(M)$

$\varphi_X^{t_1} \circ \varphi_X^{t_2} = \varphi_X^{t_1+t_2}$

($\mathbb{R} \to \mathrm{Diff}(M)$ smooth gp. homo.)

How this correspondence works:

$$\dot{\varphi}^t(\varphi^{-t}) = \underbrace{\left(\frac{\partial}{\partial t}\varphi^t\right)(\varphi^{-t}(p))}$$ $\longleftarrow\!\dashv$ φ^t one-param. subgroup

$$p \mapsto \varphi^{-t}(p) \mapsto X_{\varphi^t(\varphi^{-t}(p))}$$

In fact: " $T_{id}\,\mathrm{Diff}(M) = \Gamma(TM)$ "

$X \in \Gamma(TM)$ $\xmapsto{\text{ODE}}$ φ_X^t solves $\begin{cases} \dot{\varphi}^t = X_{\varphi(t)} \\ \varphi^0 = id_M \end{cases}$

(c.f. isotopy extension)

uniqueness of sol. $\Rightarrow$ $\varphi_X^{t+t_0} \circ (\varphi_X^{t_0})^{-1} = \varphi_X^t$

i.e. $\varphi_X^{t+t_0} = \varphi_X^t \circ \varphi_X^{t_0}$

To locally understand a "flow" φ^t it is useful to "integrate" it.

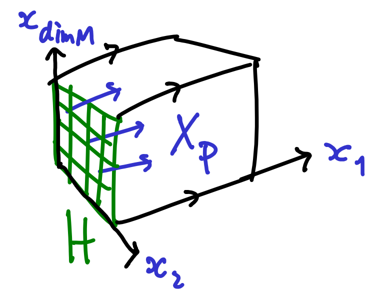
Best possible scenario:

Sometimes we can find invariant coordinates $x_1, \dots, x_{\dim M}$ s.t.

$$x_i(\varphi^t) \equiv x_i(\varphi^0) \quad i > 1, \quad |t| \text{ small},$$

i.e. s.t. φ^t is a translation of a single coord. ($\Rightarrow$ "one-dim" dynamics")

For a nonsingular X, this can be done by "flow-box coordinates":



hypersurface $H \subseteq M$ transverse to X which can be parametrised by $\psi(x_2, \dots, x_{\dim M})$

extend to a parametrisation of M:

$$\Psi(x_1, \dots, x_{\dim M}) \stackrel{\text{def.}}{=} \varphi^{x_1}(\psi(x_2, \dots, x_{\dim M}))$$

$X \neq 0$

$\Rightarrow$ dynamics is locally a linear translation $x_1 \mapsto x_1 + t$

2. The Lie bracket

What about <u>several flows</u>? To make both respect some choice of coordinates we need that the flows <u>commute</u> i.e.

$$\varphi_X^t \circ \varphi_Y^s = \varphi_Y^s \circ \varphi_X^t$$

We proceed to give the infinitesimal characterisation.

<u>Observe:</u>

$\left\{ \varphi_X^{-s} \circ \varphi_Y^t \circ \varphi_X^s \right\}_t \subseteq \mathrm{Diff}(M)$ is a one-param. subgroup $\forall s$

$$\left(\left(\varphi_X^{-s} \circ \varphi_Y^{t_1} \circ \varphi_X^s\right) \circ \left(\varphi_X^{-s} \circ \varphi_Y^{t_2} \circ \varphi_X^s\right) = \varphi_X^{-s} \circ \varphi_Y^{t_1+t_2} \circ \varphi_X^s \right)$$

φ_X^s & φ_Y^t commute $\Longleftrightarrow$ above path of one-param. subgps is independent on s, i.e. $\equiv \varphi_Y^t$

<u>Infinitesimal generator: at a point $p \in M$</u> $\quad Y_p^s \in T_pM \cong \mathbb{R}^{\dim M}$ smooth vector-space valued fcn. for each p

What is the derivative $\frac{\partial}{\partial s} Y_p^s \in T_p M \cong \mathbb{R}^{\dim M}$?

In <u>local coord^s</u> we can compute: (φ_X^s, φ_Y^t denote the flows expressed in loc. coord^s)

$$\frac{\partial}{\partial s} \tilde{Y}_p^s \Big|_{s=0} = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\left(\varphi_X^{-s} \circ \varphi_Y^t \circ \varphi_X^s(p)\right)\Big|_{\substack{t=0\\ s=0}} \overset{\text{[Chain rule]}}{=} \frac{\partial}{\partial s}\left(D\varphi_X^{-s}\left(\tilde{Y}_{\varphi_X^s(p)}\right)\right)$$

$$\text{[view v.f. as } \mathbb{R}^{\dim M} \to \mathbb{R}^{\dim M}] \quad = \left(\frac{\partial}{\partial s} D\varphi_X^{-s}\right)\tilde{Y}_{\varphi_X^0(p)} \overset{\text{[prod. rule]}}{+} \underbrace{D\varphi_X^{-0}}_{\text{id}}\left(D_p\tilde{Y}(\tilde{X}_p)\right)$$

$$\text{[change order of differentiation]} \quad = \left(D\left(\frac{\partial}{\partial s}\varphi_X^{-s}\right)\right)\tilde{Y}_{\varphi_X^0(p)} + D_p\tilde{Y}(\tilde{X}_p)$$

$$= \left(D_p\tilde{Y}\right)\tilde{X}_p - \left(D_p\tilde{X}\right)\tilde{Y}_p$$

Obs: this computation <u>only</u> makes sense in loc. coord^s

<u>Exercise. 35.)</u> Show that $\boxed{[X,Y]_p \overset{\text{def.}}{=} \frac{\partial}{\partial s} Y_p^s\Big|_{s=0}}$ called the <u>Lie derivative of Y in the direction X</u> defines a Lie algebra

$$[\cdot,\cdot] : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM), \text{ i.e:}$$

- bilinear
- $[X,Y] = -[Y,X]$
- $[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$ (Jacobi identity)

(This Lie alg. corr. to the "Lie group" Diff(M))

3. The de Rham complex.

Let $\mathrm{Hom}(TM, \mathbb{R})$, $\mathrm{Hom}(TM^{\wedge k}, \mathbb{R})$, etc. denote the family of antisymmetric k-multilinear maps $\underbrace{T_{pt}M \times \dots \times T_{pt}M}_{k} \to \mathbb{R}$ "parametrised by $pt \in M$"

In fact, there are also vector bundles

$k=1$: T^*M, etc.

$\Gamma(\mathrm{Hom}(TM^{\wedge k}, \mathbb{R}))$ are the global sections,

or more concretely:

- $\Gamma(T^*M) \subseteq C^\infty(TM, \mathbb{R})$ linear in each $T_{pt}M$
- $\Gamma(\mathrm{Hom}(TM^{\wedge k}, \mathbb{R})) \subseteq C^\infty(TM^{\wedge k}, \mathbb{R})$ ....

There exists an (infinite dimensional) canonical chain complex

$$C^\infty(M,\mathbb{R}) \xrightarrow{d^0} \Gamma(T^*M) \xrightarrow{d^1} \Gamma(\mathrm{Hom}(TE^{\wedge 2}, \mathbb{R})) \to \dots \qquad \boxed{d \circ d = 0}$$

1-forms

2-forms = antisymmetric bilinear maps of tangent vectors

d is the "exterior derivative":

- The exterior derivative of a function is the usual derivative:

$$df \overset{\text{def.}}{=} \widetilde{Tf}, \quad \text{i.e. } Tf(X_p) = \{f(p)\} \times \widetilde{Y}_p \in T\mathbb{R} \qquad TM \xrightarrow{Tf} T\mathbb{R} \to \mathbb{R}$$

"eats" tangent vectors

$$df(X_p) = \widetilde{Y}_p \in \mathbb{R}$$

df

Consider $df(X)$ as the "partial derivative of f in the direction X"

sometimes we write $X(f) \overset{\text{def.}}{=} df(X)$.

- For an 1-form η we can define its <u>exterior derivative</u>

$$d\eta(U_p, V_p) \overset{\text{def.}}{=} d(\underbrace{\eta(V)}_{M\to\mathbb{R}})U_p - d(\underbrace{\eta(U)}_{M\to\mathbb{R}})V_p - \eta[U,V]_p$$

after a choice of extensions of U_p & V_p to vector fields.

(clearly antisymmetric!)

<u>Exercise 36.)</u> Show that the R.H.S above is indep. of the choice of extension.

Hint: first establish

$$\boxed{[U, V + fW] = [U,V] + f[U,W] + (df(U))\cdot W} \qquad f: M \to \mathbb{R}$$