For <u>local</u> coordinates $\varphi : \mathbb{R}^n \hookrightarrow M$ $\quad \frac{\partial}{\partial x_i} \overset{\text{def.}}{=} T\varphi \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{:th entry}$

(locally defined <u>coord. vector field</u>)

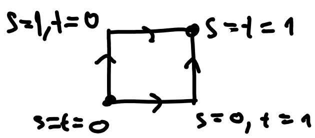
Since $\varphi^t_{\frac{\partial}{\partial x_i}}(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, x_i + t, x_{i+1}, \dots, x_n)$

$$\Rightarrow \varphi^t_{\frac{\partial}{\partial x_i}} \circ \varphi^s_{\frac{\partial}{\partial x_j}} = \varphi^s_{\frac{\partial}{\partial x_j}} \circ \varphi^t_{\frac{\partial}{\partial x_i}} \Rightarrow \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right] = 0$$

where defined

makes sense <u>locally</u>

i.e.



<u>**Cor. 33**</u> For $f \in C^\infty(M, \mathbb{R})$ we have $\underbrace{d^2 f}_{\text{2-form } \in \Gamma(\mathrm{Hom}(TM^{\wedge 2}; \mathbb{R}))} = 0$.

<u>Proof.</u> Use formula for $d(df)$ with some locally defined <u>coordinate vector fields</u> $\frac{\partial}{\partial x_i}$ & $\frac{\partial}{\partial x}$ $\left(\Rightarrow \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right] = 0 \right)$.

$$d^2 f\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = d\left(df\left(\frac{\partial}{\partial x_j}\right)\right)\left(\frac{\partial}{\partial x_i}\right) - d\left(df\left(\frac{\partial}{\partial x_i}\right)\right)\left(\frac{\partial}{\partial x_j}\right)$$

$$= d\left(\frac{\partial f}{\partial x_j} \right)\left(\frac{\partial}{\partial x_i} \right) - d\left(\frac{\partial f}{\partial x_i} \right)\left(\frac{\partial}{\partial x_j} \right)$$

$$= \frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i} = 0 .$$

Since $\frac{\partial}{\partial x_i}$ form a basis $\Rightarrow d^2 f$ (locally) $= 0$ $\qquad \square$

<u>As a consequence</u>: $df[X, Y] = d(df(Y))(X) - d(df(X))(Y)$

4. Integration of k-forms

There is no natural way to integrate a function on a mfd. (unless the additional choice of a metric is made).

The integral depends on the area units induced by loc. coord^s, and different coordinates thus give a different result.

But we can integrate k-forms over compact <u>oriented</u> k-dim^l submanifolds Σ^k with possibly nonempty boundary. (Or more generally: "smooth k-chains".)

Definition of $\int_{\Sigma^k} \eta$ for $\Sigma^k \subseteq M$ & $\underbrace{\eta \in \Gamma(\mathrm{Hom}(TM^{\wedge k}), \mathbb{R}))}_{k\text{-form}}$

For a subset $\varphi(\mathcal{U}) \subseteq \Sigma^k$ that can be parametrised by

$$\mathbb{R}^k \supseteq \overset{\text{(nice cpct)}}{\mathcal{U}} \overset{\varphi}{\hookrightarrow} \Sigma^k$$

we define (& compute) this integral by the formula

$$\int_{\mathcal{U}} \underbrace{\eta\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right)(x_1, \dots, x_k)}_{\mathcal{U} \to \mathbb{R} \text{ smooth}} dx_1 \dots dx_k$$

<u>In general</u>: divide Σ^k into smaller pieces covered by coordinate charts.

The reason why this is <u>invariant</u> under orientation preserving reparametrisations is the change of variables formula for integrals and the following transformation rule:

> **Fact:** α skew-symm. k-form on a k-dim. vector space V
> $\bar{e}_1, \dots, \bar{e}_k$, $\bar{f}_1, \dots, \bar{f}_k$ two bases $\Psi_{i,j} = \langle \bar{e}_i, \bar{f}_j \rangle_{\bar{e}}$ base change mtx.
> then $\alpha(\bar{f}_1, \dots, \bar{f}_k) \overset{(*)}{=} \alpha(\bar{e}_1, \dots, \bar{e}_k) \cdot \det \Psi$

<u>Exercise 37.)</u> Show $(*)$ when $k = 2$.

For different coordinates $\underset{\nwarrow\ \mathbb{R}^n \to \mathbb{R}^n}{\psi \circ \varphi} : \overset{\subseteq \mathbb{R}^n}{\varphi^{-1}(U)} \hookrightarrow \Sigma^k$ inducing the <u>same orientation</u> we compute:

$$\int_{\psi^{-1}(U)} \eta\left(\frac{\partial}{\partial \tilde{x}_1}, \dots, \frac{\partial}{\partial \tilde{x}_k}\right)(\tilde{x}_1, \dots, \tilde{x}_k)\, d\tilde{x}_1 \dots d\tilde{x}_k =$$

Jacobian > 0 by orientation preserving prop.

$$\overset{(*)}{=} \int_{\psi^{-1}(U)} \eta\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right)(\tilde{x}_1, \dots, \tilde{x}_k) \det\left[\frac{\partial \psi^i}{\partial \tilde{x}_j}\right] d\tilde{x}_1 \dots d\tilde{x}_k =$$

$$\overset{\text{ch. of var.}}{=} \int_U \eta\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right)(x_1, \dots, x_k)\, dx_1 \dots dx_k .$$

For $\Sigma^2 \hookrightarrow M$ & $\eta \in \Gamma(\mathrm{Hom}(TM,\mathbb{R})) = \Gamma(T^*M)$:

<u>**Thm. 34**</u> (Stokes')

$$\int_{\Sigma^2} \overbrace{d\eta}^{\text{2-form}} = \int_{\partial\Sigma^2} \eta \leftarrow \text{1-form}$$

$\partial\Sigma^2 \leftarrow$ 1-dim^{l} mfd

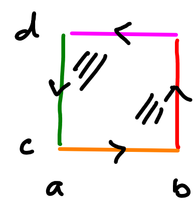
<u>Proof</u> When Σ^2 is parametrised by a rectangle

$$\underset{x}{u : [a,b]} \times \underset{y}{[c,d]} \to M$$

$$\text{L.H.S.} = \int_a^b\int_c^d d\left(\eta\left(\frac{\partial}{\partial y}\right)\right)\left(\frac{\partial}{\partial x}\right) - d\left(\eta\left(\frac{\partial u}{\partial x}\right)\right)\left(\frac{\partial}{\partial y}\right) - \eta\Big[\overbrace{\cancel{\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}}}^{\equiv 0 \text{ since coord. v.f.}}\Big]\, dx\, dy$$

$$= \int_a^b\int_c^d \frac{\partial}{\partial x}\eta\left(\frac{\partial}{\partial y}\right) - \frac{\partial}{\partial y}\eta\left(\frac{\partial u}{\partial x}\right) dx\, dy$$

$$\left[\begin{matrix}\text{fund. thm.}\\ \text{of calculus}\end{matrix}\right] = \int_c^d \eta\left(\frac{\partial}{\partial y}(b,y)\right) - \eta\left(\frac{\partial}{\partial y}(a,y)\right) dy$$



$$- \int_a^b \eta\left(\frac{\partial}{\partial x}(x,d)\right) - \eta\left(\frac{\partial}{\partial x}(x,c)\right) dx$$

$$= \text{R.H.S.} \qquad \square$$

In particular $\int_{\Sigma^2} d\eta = 0$ when Σ^2 is <u>closed</u>.

5. Lie algebra of a Lie group

G Lie group

$m: G \times G \to G$ smooth multiplication

$l_g: G \to G$, $h \mapsto gh$ smooth mult. from the left.

$\Gamma(TX) \supseteq \mathfrak{g}$ the left G-equivariant vector fields $\longleftrightarrow$ left G-equiv. one-parameter subgroups of $\mathrm{Diff}(G)$

$\Longleftrightarrow$ one param. subgroups $\subseteq G$

$$\varphi_X^t(g) = g \cdot \varphi_X^t(e)$$

$\mathfrak{g} \cong T_eG$ since X equiv.

$\Longleftrightarrow$

$$X_g = Dl_g \cdot X_e \in T_gG$$

$l_g: G \xrightarrow{\cong} G$

mult. from left by $g \in G$

Since $\varphi_X^{-s} \circ \varphi_Y^t \circ \varphi_X^s(g) = g \cdot \varphi_X^{-s} \circ \varphi_Y^t \circ \varphi_X^s(e)$ is G-equiv. $\Rightarrow$

preserves $\mathfrak{g}$

$$(\mathfrak{g}, [\_\,,\_]) \subseteq (\Gamma(TG), [\_\,,\_])$$

finite-dim. Lie subalgebra of $\Gamma(TG)$ (infinite-dim')

In the case of classical matrix Lie groups:

$$X_e \in T_eG \in \mathrm{Mat}_{n\times n} \Rightarrow \varphi^t_X(e) = e^{tX_e} \in G \subseteq Gl_n$$ one-param. subgroup

Exercise 38.) Show that in this case

$$[X,Y]_e = \underbrace{X_e \cdot Y_e - Y_e \cdot X_e}_{\text{commutator of matrices}}, \quad X, Y \in \mathfrak{g}$$

The adjoint representation

$$\varkappa : G \longrightarrow \mathrm{Diff}(G)$$

$$g \mapsto \varkappa_g, \qquad \varkappa_g(\sqcup) = g \cdot \sqcup \cdot g^{-1}$$

this is a Lie group homomorphism

vector space!

$$\mathrm{Ad}: G \longrightarrow Gl(\mathfrak{g}) \cong Gl(T_eG) \subseteq \mathrm{End}(T_eG)$$ finite-dim. representation

$$g \mapsto \left[\underset{\uparrow \atop e\mathfrak{g}}{Y} \mapsto \underbrace{\frac{\partial}{\partial t} \varkappa_g \circ \varphi^t_Y \circ \varkappa_g^{-1}}_{= T_e\varkappa_g(Y_e) \in T_eG \text{ (by the chain rule)}} \right]$$

Obs: $$\varkappa_g \circ \varphi^t_Y \circ \varkappa_g^{-1}(h) = g \cdot \varphi^t_Y(g^{-1}hg)$$

$$= h \cdot g \cdot \varphi^t_Y(e) \cdot g^{-1}$$

$$= h \cdot \varkappa_g \circ \varphi^t_Y \circ \varkappa_g^{-1}(e) \Rightarrow$$ left-equivariant one param. subgroup for $\forall g \in G$.

$$\mathrm{ad} := d\,\mathrm{Ad}|_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$$

$$X \longmapsto \mathrm{ad}_X(\_\!\_) = [X, \_\!\_]$$

morphism of Lie algebras by Jacobi id. (End is endowed w. commutator as Lie bracket)

Proof. $\varkappa_{\varphi_X^s(e)} \circ \varphi_Y^t \circ \varkappa^{-1}_{\varphi_X^s(e)}(g) = \varphi_X^s(e) \cdot \varphi_Y^t\left(\varphi_X^{-s}(e) \cdot g \cdot \varphi_X^s(e)\right) \cdot \varphi_X^{-s}(e)$

$$= \varphi_Y^t\left(\varphi_X^s(g)\right) \cdot \varphi_X^{-s}(e)$$

$$= \varphi_X^{-s}\left(\varphi_Y^t\left(\varphi_X^s(g)\right)\right)$$

Recall the definition of the Lie derivative (c.f. previous lecture)

$$\mathrm{ad}_Y(X) = \frac{\partial^2}{\partial s \partial t} \varphi_X^{-s} \circ \varphi_Y^t \circ \varphi_X^s = [X, Y]$$

$\square$