6. The Cartan one-form

The de Rham complex of "lie-algebra valued forms"

$$C^\infty(M,\mathfrak{g}) \xrightarrow{d} \Gamma(\mathrm{Hom}(TM,\mathfrak{g})) \xrightarrow{d} \Gamma(\mathrm{Hom}(TM^{\wedge 2},\mathfrak{g}))$$

<u>The Cartan one-form</u> $\vartheta : TG \to T_eG \cong \mathfrak{g} \in \Gamma(\mathrm{Hom}(TG),\mathfrak{g}))$

$$X_g \mapsto T\ell_{g^{-1}}(X_g)$$

ϑ is <u>left G-invariant.</u>

<u>Pf.</u> $\vartheta(T\ell_h(X_g)) = T\ell_{(hg)^{-1}}(T\ell_h(X_g))$

$$[\text{chain rule}] = T\ell_{g^{-1}} \circ \underbrace{T\ell_{h^{-1}} \circ T\ell_h}_{\mathrm{Id} \text{ by chain rule}}(X_g) = \vartheta(X_g)$$

$f : M \to N$ smooth, $\eta \in \Gamma(T^*N)$ one-form

$\Rightarrow$ $f^*\eta \overset{\text{def.}}{=} \eta \circ Tf \in \Gamma(T^*M)$ again a one-form (the <u>pullback</u>)

the above thus says: $\boxed{\ell_g^* \vartheta = \vartheta \qquad (\mathrm{LI})}$

For multiplication from the right:

$$\vartheta \circ Tr_g(X_e) \overset{(\text{def})}{=} T\ell_{g^{-1}} \circ Tr_g(X_e) \overset{(\text{ch. rule})}{=} Tc_{g^{-1}}(X_e) \overset{(\text{def.})}{=} Ad_{g^{-1}}(X_e)$$

$[r_g \circ \ell_h = \ell_h \circ r_g] \Rightarrow \boxed{r_g^* \vartheta = Ad_{g^{-1}} \circ \vartheta \qquad (\mathrm{RI})}$

The <u>Maurer - Cartan equation</u> for $\vartheta \in \Gamma(\mathrm{Hom}(TG, \mathfrak{g}))$

$$\boxed{d\vartheta(v_p, w_p) + [\vartheta(v_p), \vartheta(w_p)]_{\mathfrak{g}} = 0 \quad (M\text{-}C)}$$

<u>Proof.</u> For <u>left G-equiv. extensions</u> V & W of v_p & w_p:

$$d\vartheta(v_p, w_p) = d\,(\underbrace{\vartheta(W)}_{\equiv W})\, v_p - d(\underbrace{\vartheta(V)}_{\equiv V})\, w_p - \vartheta[V, W]_p \leftarrow \text{in } \Gamma(TG)$$

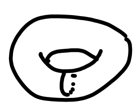
$$= 0 - 0 - [\vartheta(v), \vartheta(w)]_{\mathfrak{g}}$$

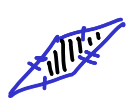
$\square$

(1.) ϑ provides a <u>trivialisation</u> $TG \cong G \times \mathfrak{g}$

$$X_g \mapsto (g, \vartheta(X_g))$$

<u>Warning</u> unless $[\_, \_]_{\mathfrak{g}} = 0$, (M-C) implies that the trivialisation is not given by coordinate vector fields.
(Compare $(S^1)^n$ abelian & $SU(2) \cong S^3$ nonabelian)

(2.) The only compact conn. surface which admits the structure of a Lie group is $\mathbb{T}^2$  by (1.) . This will follow from Gauß-Bonnet (to come later).

(3.) $\mathbb{R}^2 / \underbrace{2\text{-dim lattice}}_{\mathbb{Z}^2 \hookrightarrow \mathbb{R}^2}$ =  $\cong \mathbb{T}^2$. $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ <u>global</u> coord. v.f. invariant under $\mathbb{T}^2$-action

$\Rightarrow (\mathfrak{g}, [\_, \_]) \cong (\mathbb{R}^2, 0)$ (abelian group!)

(4.) For <u>any</u> Lie group structure on $\mathbb{T}^2$:

$$\overset{\text{(Stokes!)}}{0 = \int_{\mathbb{T}^2} d\theta} \overset{(*)}{=} \int_{\mathbb{T}^2} d\theta(X,Y) \cdot \underbrace{\det \Phi}_{>0}\, dx\, dy \overset{\text{(M-C)}}{=} \int_{\mathbb{T}^2} -[X,Y]_{\mathfrak{g}} \cdot \underbrace{\det \Phi}_{>0}\, dx\, dy$$

X, Y: left inv. oriented basis of $\mathfrak{g}$; Φ: suitable basis change matrix

$\Rightarrow [\sqcup,\sqcup] = 0 \Rightarrow$ abelian $\overset{(\dots)}{\Rightarrow} \cong \mathbb{R}^2/\text{lattice}$

(5.) $H \overset{\iota}{\hookrightarrow} G$ closed Lie subgroup $\Rightarrow \theta_H = \theta_G \circ T\iota$

(6.) If $H^1_{dR}(G) = 0$ and G compact, then $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. ($\Rightarrow$ not solvable!)

$H^1_{dR}(G) = \mathrm{Hom}(H_1(G,\mathbb{Z}), \mathbb{R})$, $H_1(G,\mathbb{Z}) = \pi_1(G)/\text{comm.}$

e.g. $G = SU(2)$

<u>Proof.</u> Assume by contradiction that $[\mathfrak{g},\mathfrak{g}] \subsetneq \mathfrak{g}$.

Take a projection $\pi: \mathfrak{g} \twoheadrightarrow \mathbb{R}v \subseteq \mathfrak{g}$ with $\ker \pi \supseteq [\mathfrak{g},\mathfrak{g}]$

$\eta := \pi \circ \theta \in \Gamma(T^*G)$ is a one-form which is:

- closed by (M-C), i.e. $d\eta = d\pi \circ \theta = \pi \circ d\theta \overset{\text{(M-C)}}{=} 0$
- not exact, i.e. $df \neq \eta$ for any $f \in C^\infty(G,\mathbb{R})$,

since $\pi(\theta(Tl_g(v))) = \pi(v) = v$ this is a <u>nowhere vanishing form</u> (df vanishes at critical points)

7. Connections on principal bundles

<u>Principal bundle:</u> $G \hookrightarrow \underset{\circlearrowleft r_g}{E} \xrightarrow{\pi} E/G = B$

Now we consider <u>smooth</u> principal bundles, i.e. G Lie group, $E \circlearrowleft G$ smooth manifold, $B = E/G$ smooth.

E can be seen as a family of groups G, we want a "family of Cartan forms." The correct notion is the following, often called an <u>Ehresmann-connection:</u>

Lie alg. of G

Def. A <u>connection on E</u> is $\omega \in \Gamma(\mathrm{Hom}(TE, \mathfrak{g}))$ s.t.

(A1) $\omega \circ T\iota_{pt} : TG \longrightarrow \mathfrak{g}$ is the Cartan form ϑ on G,

i.e. $\iota_{pt}^* \omega = \vartheta$, along any $\iota_{pt} : \begin{matrix} G \hookrightarrow E \\ g \mapsto pt \cdot g \end{matrix}$. (coincides w. θ_G along the fibres)

(A2) $r_g^* \omega \; (= \omega \circ Tr_g) = Ad_{g^{-1}} \circ \omega \quad r_g : \begin{matrix} E \to E \\ pt \mapsto pt \cdot g \end{matrix}$ (invariant when G is abelian!)

<u>Exercise 39.)</u> Show that there is a bij. corr. between connections as above and decompositions into horizontal & vertical tangent spaces $T_{pt}E = \underset{\text{horiz.}}{H_{pt}} \oplus \underset{\text{vert.}}{\mathrm{im}\, T_e \iota_{pt}}$ s.t. $H_{pt \cdot h} = Tr_h(H_{pt})$ via the assignment $H = \ker \omega$.

Ex. The trivial connection ω_{triv} on the trivial principal bundle $E = G \times B$ is given by $\omega_{triv} \overset{def.}{=} \vartheta_G \circ T\,pr_G$ for the canonical projection $pr_G : G \times B \twoheadrightarrow G$. ($\exists$ only for trivial bundles!)

(A1) $$\omega_{triv} \circ T\iota_{(g,p)} \overset{(def)}{=} \vartheta \circ T\,pr_G \circ T\iota_{(g,p)} \overset{(ch. rule)}{=} \vartheta \circ T\ell_g \overset{(LI)}{=} \vartheta$$

(A2) $$\omega_{triv} \circ Tr_g \overset{(def)}{=} \vartheta \circ T\,pr_G \circ T\underbrace{r_g}_{\text{on } E} \overset{(ch. rule)}{=} \vartheta \circ T\underbrace{r_g}_{\text{on } G} \circ T\,pr_G$$

$$\overset{(RI)}{=} Ad_g \circ \vartheta \circ T\,pr_g \overset{(def.)}{=} Ad_g \circ \omega_{triv}$$

Often one is interested in the space of connections up to Gauge transformation, i.e.

$$\omega \sim \Psi^* \omega \quad \text{for any } \Psi \in \mathcal{G}(E) \quad \text{(c.f. Lecture 7)}$$

Rmk In physics E describes the choice of phase/gauge for for certain waves (e.g. Electro-Magnetism: electric waves, $G = S^1$). Connections are potentials of other fields that couple to the waves (e.g. magnetic potentials). Physical significant only mod. $\mathcal{G}(E)$

A computation in a trivial bundle (E is locally trivial!)

ω connection on $G \times B$. Obs: $T(G \times B) = TG \times TB$

We conclude from (A1) & (A2) that

$$\omega = \omega_{triv} + A, \quad A \in \Gamma(\mathrm{Hom}(TE, \mathfrak{g}))$$

A depends on ω & trivialisation!

where $A|_{TG} \overset{(A1)}{=} 0$ & $A(0_g, {}_p) \overset{(A2)}{=} Ad_{g^{-1}} A(0_e, Y_p)$

$$\Rightarrow A(0_{hg}, Y_p) = Ad_{(hg)^{-1}} A(0_e, Y_p) = Ad_{(hg)^{-1}} \circ Ad_g (0_g, Y_p)$$
$$= Ad_{g^{-1}h^{-1}g} A(0_g, {}_p)$$

Any $\Psi \in \mathcal{G}(G \times B)$ is of the form

$$(g, p) \overset{\Psi}{\longmapsto} (\psi(p) \cdot g, p), \qquad \psi : B \to G.$$

$$\psi^*(r_g^* \theta) = Ad_{g^{-1}}(\psi^* \theta) \quad \text{for } \psi^*\theta \in \Gamma(\mathrm{Hom}(TB, \mathfrak{g}))$$

By the above we conclude that

$$\boxed{\begin{aligned} \Psi^* \omega &= \omega_{triv} + A_\psi \qquad\qquad (GT) \\ A_\psi(X_g, Y_p) &= Ad_{g^{-1}}(\psi^*\theta)(Y_p) + Ad_{g^{-1}\psi(p)^{-1}g} \circ A(0_g, Y_p) \end{aligned}}$$

This also explains the dependence on the local triv.