Prop. 35 The connections on a principal bundle form a nonempty affine (and thus contractible) space.

Proof.

Global convex interpolation gives:

$(1-t)\,\omega_0 + t\,\omega_1$ $\underbrace{\text{are all connections}}_{\text{(A1) \& (A2) are fibrewise lin. cond}}$ $\Rightarrow$ affine space

More generally, for $\sigma : B \xrightarrow{C^\infty} [0,1]$ we can perform a fibrewise convex interpolation:

$(1-\sigma)\cdot\omega_0 + \sigma\cdot\omega_1$ is a connection

We can now use the local existence of connections, e.g. ω_{triv} determined by the choice of a local trivialisation

$$\pi^{-1}(\mathcal{U}) \cong G \times \mathcal{U}$$

together w. partition of unity argument to show existence of a globally defined connection.

Exercise 40.) Give the details of the part. of unity argument.

□

S^1-bundles

$\mathbb{C} \supseteq S^1 = U(1) = SO(2)$ abelian $\Rightarrow$ $Ad \equiv Id_{\mathfrak{g}}$, $ad \equiv 0$

$\exp(i \cdot \_\_): \mathbb{R}/2\pi\mathbb{Z} \xrightarrow{\cong} S^1$ (even though no global coord)

global "coord. v.f."

Obs: $T(\mathbb{R}/2\pi\mathbb{Z}) = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$, $\frac{\partial}{\partial t} \in \Gamma(T(\mathbb{R}/2\pi\mathbb{Z}))$

$\frac{\partial}{\partial \theta} \overset{\text{def.}}{=} T\exp(i\_\_)\,\frac{\partial}{\partial t}$ ($\frac{\partial}{\partial \theta}$ infinitesimal rot: 1 rad / time unit)

a choice of basis

dep^s on "length" of S^1; here 2π

$\mathfrak{g}_{S^1} = \mathbb{R} \cdot \frac{\partial}{\partial \theta} \cong \mathbb{R}$ w. trivial bracket

$r \cdot \frac{\partial}{\partial \theta} \mapsto r$

The Cartan form: $\vartheta_{S^1}\left(\left(r \cdot \frac{\partial}{\partial \theta}\right)_{pt}\right) = r \cdot \frac{\partial}{\partial \theta}$

$\in T_{pt} S^1$

Under the above identification $\mathbb{R} \cdot \frac{\partial}{\partial \theta} \cong \mathbb{R}$ we thus get an identification

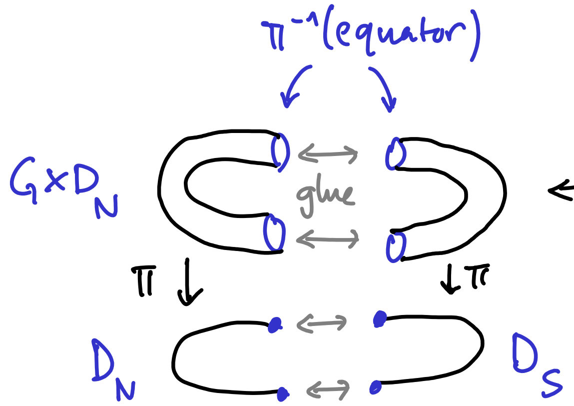
$\theta \in \Gamma(T^* S^1)$ a <u>real-valued</u> one-form

$\theta(X_{pt})$ = "rad / time of infinitesimal rot. X_{pt} at $pt \in S^1$"

For $f \in C^\infty(M, S^1)$ we write

$df \overset{\text{def.}}{=} f^*\theta \in \Gamma(T^*M)$ a <u>closed</u> (⚠ possibly non-exact) real-valued one-form on M!

Recall the classification of S^1-bundles from Lecture 7:



$\leftarrow \pi^{-1}(D_S) \cong G \times D_S$ since D_S contractible

$B = S^1 = D_N \cup D_S$

E produced by gluing two trivial bundles defined on the hemispheres D_N, D_S, along the equator by $\alpha : S^0 \to S^1$ $\leftarrow$ choice of rot. when gluing

Since α unique / homotopy $\Rightarrow$

$$\begin{array}{l} E \cong S^1 \times B \\ \quad \downarrow \pi = pr_B \\ \quad B = S^1 \end{array}$$

<u>Exercise 41.)</u> Show that a connection on $E = S^1 \times S^1$ can be canonically identified with some $\omega \in \Gamma(T^*E)$ that satisfies

(A1) $\omega\left(\frac{\partial}{\partial\theta}\right) = 1$ & (A2) $r_g^* \omega = \omega$.

<u>Exercise 42.)</u> Show that $\mathcal{G}(S^1 \times B) \overset{(G \text{ abelian})}{=} C^\infty(B, G)$ $\quad B = S^1$

$\Psi = r_\psi \leftrightarrow \psi$

Further, for $\Psi = r_\psi$ we have: $\Psi^* \omega = \omega + d\psi$ $\quad \psi^*\theta$

8. Parallel transport

A connection gives the infinitesimal notion of "parallelability" (or horizontality).

Def. A section $\sigma: M \to E$ along a submanifold $M \overset{\iota}{\hookrightarrow} B$ (i.e. $\iota = \pi \circ \sigma$) is parallel w.r.t. ω if $\iota^* \omega = 0$.

- When $\dim M = 1$ being parallel is controlled by an ODE; there always exist unique parallel sections over $[0,1] \overset{\iota}{\hookrightarrow} B$ coinciding w. a given $pt \in p^{-1}(\iota(0))$ above $\iota(0)$ (see below).

- When $\dim M \geqslant 2$ parallel sections are the solutions of a PDE; there may be no parallel sections in this case.

We define the parallel transport along the path $[0,1] \overset{\iota}{\hookrightarrow} B$ to be $\prod_{\omega, \iota} : \pi^{-1}(\iota(0)) \to \pi^{-1}(\iota(1))$ constructed as follows

(Can also be defined for arbitrary smooth paths)

Let $X \in \Gamma(T(\overbrace{\pi^{-1}(\iota[0,1])}^{\text{"}G\times\iota[0,1]\text{"}\subseteq E}))$ be the v.f. det. by

$$\boxed{\omega(X)=0 \;\&\; T\pi(X)=\frac{\partial}{\partial t}}$$

horizontal — lifts the coord. v.f. on $[0,1]\ni t$

$\Pi_{\omega,\iota}(pt) \overset{\text{def.}}{=} \sigma(1)$ where $\sigma:[0,1]\longrightarrow E$ solves the ODE

$$\begin{cases} \frac{d}{dt}\sigma(t) = X_{\sigma(t)} \\ \sigma(0) = pt \in \pi^{-1}(\iota(0)) \end{cases}$$

<u>Exercise 43.)</u> Show that 1.) $\pi\circ\sigma(t)=\iota(t)$ (i.e. σ is a section)

2.) $\Pi_{\omega,\iota}(pt\cdot g)=\Pi_{\omega,\iota}(pt)\cdot g$ (<u>hint</u>: use Exercise 39.))

Parallel sections need not exist along $S^1 \overset{\iota}{\hookrightarrow} B$!
The solution σ might develop a <u>discontinuity</u> when approaching the endpoint = starting point
$\exp(i\cdot 2\pi)$ $\exp(i\cdot 0)$

The <u>monodromy</u> $\Pi_{\omega,\iota,t}: \pi^{-1}(\iota(\exp(it))) \circlearrowleft$ is the "parallel transport from $\exp(i\cdot t)$ to $\exp(i(t+2\pi))$".

$\cong G$

We can use the right G-action to identify the fibre $pt \in \pi^{-1}(\iota(\exp(i\theta))) = pt\cdot G \cong G$ with a G-orbit

$(\iota_{pt} : G \hookrightarrow pt\cdot G)$

(Exercise 43.) $\Rightarrow$ $\Pi_{\omega,\iota,t}(pt\cdot g) = \underbrace{pt\cdot h_{\omega,\iota,t,pt}}_{\Pi_{\omega,\iota,t}(pt)}\cdot g$

<u>Exercise 44.)</u> Show:

- Dependence on pt: $h_{\omega,\iota,t,pt\cdot h} = h^{-1}\cdot h_{\omega,\iota,t,pt}\cdot h$
- Dependence on t: $h_{\omega,\iota,t,pt} = h_{\omega,\iota,t+\delta,\Pi(pt)}$

where Π = "parallel transport" from t to $t+\delta$

<u>Obs</u> $\Pi_{\omega,\iota(1-t)} = \Pi_{\omega,\iota}^{-1}$ by uniqueness of sol. of ODEs.

(Exc. 44)

$\Rightarrow$ "monodromy along $S^1 \hookrightarrow B$" is well-defined when G is <u>abelian</u>, and it can be identified with some $h \in G$

E.g. when $G = S^1 \Rightarrow h$ = rotation

<u>Exc.</u> ω_{triv} on $E = G\times B$ has trivial monodromies (and parallel transports).

Exercise 45.)* Show that when $E = \underbrace{S^1}^{G} \times \underbrace{S^1}^{B}$, then

$$\omega \mapsto \Pi_{\omega, id_{S^1}, 1} \in S^1 \subseteq C^\infty(\underset{G}{S^1}, \underset{G}{S^1}) \qquad g \mapsto \ell_g$$

(the monodromy)

is a bijection on the Gauge equivalence classes.