Prop. 37 Let $\iota : [0,1] \hookrightarrow B$ be a smooth path, $\sigma : [0,1] \to E$ a section along ι (i.e. $\pi \circ \sigma = \iota$), and $e^{X(t)} : [0,1] \to G$, $\theta\left(\frac{d}{dt} e^{X(t)}\right) = X(t)$, $e^{X(0)} = e$, a path in G.

Then:
$$\underbrace{\omega\left(\frac{d}{dt}\left(\sigma(t) \cdot e^{X(t)} \cdot g\right)\right)}_{\left(r_{e^{X(t)}\cdot g} \circ \sigma\right)^{*} \omega\left(\frac{\partial}{\partial t}\right)} = \mathrm{Ad}_{e^{X(t)}\cdot g}\, \omega\underbrace{\left(\frac{d}{dt}\sigma\right)}_{\sigma^{*}\omega\left(\frac{\partial}{\partial t}\right)} + \mathrm{Ad}_g X(t)$$

Proof.
$$\frac{d}{dt} \underbrace{r_{e^{X(t)}\cdot g}}_{\triangle\!\!\!! \text{ order } r_g \circ r_{e^{X(t)}}} \circ \sigma(t) \overset{\text{(ch. rule)}}{=} T r_{e^{X(t)}\cdot g}\left(\frac{d}{dt}\sigma(t)\right) + \underbrace{T r_g \circ T \iota_{\sigma(t)}\left(\frac{d}{dt} e^{X(t)}\right)}_{r_{e^{X(t)}\cdot g}(\sigma) = r_g \circ \iota_\sigma\left(e^{X(t)}\right)}$$

Then we use (A1) & (A2) $\square$

Consequence: Knowing $\sigma^{*}\omega$ for one section ω can be used to compute $\tilde{\sigma}^{*}\omega$ for any other $\left(\tilde{\sigma} = r_{e^{X(t)}\cdot g} \circ \sigma\right.$ for some g & $X(t)$ by transitivity of right G-action$\left.\right)$.

In particular: σ & $r_{e^{X(t)}g} \circ \sigma$ parallel $\Rightarrow X(t) \equiv 0$.

"Trivial connections" can induce monodromy (also called holonomy)

$\tilde{H}$ discrete group which acts freely on $\tilde{B}$ from right. $\tilde{B} \circlearrowleft \tilde{H}$

$\varphi : \tilde{H} \twoheadrightarrow H \leq G$ discrete subgroup

$$G \hookrightarrow (E, \omega) \overset{\text{def.}}{=} (G \times \tilde{B}, \omega_{triv}) / \tilde{H} \twoheadrightarrow \tilde{B}/\tilde{H} = B$$

$\tilde{h} \cdot (g, \tilde{b}) = (\varphi(\tilde{h}) \cdot g, \tilde{b} \cdot \tilde{h}^{-1})$

has nontrivial monodromies although ω is "locally $\cong \omega_{triv}$"
(in $\dot{B}$)

$$\begin{tikzcd}
G \times \tilde{B} \arrow[r, two heads] \arrow[d, "\tilde{\pi} = pr_{\tilde{B}}"] & E \arrow[d, "\pi"] \\
\tilde{B} \arrow[r, two heads] & B
\end{tikzcd}$$

Ex $\tilde{B} \to B$ universal cover, $\tilde{B} \circlearrowleft \tilde{H} = \pi_1(B)$
$G = Gl(V)$
φ representation $\pi_1(B) \twoheadrightarrow H \leq Gl(V)$

One says that (E, ω) is flat if it is isom. to a quotient of the trivial bundle with the trivial connection as above

9. Curvature

The obstruction to being isomorphic to a flat bundle, or more generally, to find "parallel" sections along a surface, is the curvature two-form

$$\boxed{\Theta_\omega(X,Y) \overset{\text{def.}}{=} d\omega(X,Y) + [\omega(X),\omega(Y)] \in \Gamma(\mathrm{Hom}(TE^{\wedge 2},\mathfrak{g})).}$$

Prop 38. $\Theta_\omega(X,Y)=0$ if one of $X,Y \in \mathrm{im}\, T\iota_{pt}$; in particular, Θ_ω is determined by its restriction to $H\wedge H \subseteq TE\wedge TE$.

Exercise 46.) Prove Prop. 38 in the case $G=S^1$.

Prop 39. If $X,Y \in H_{pt}$, then

$$\Theta_\omega(Tr_g X, Tr_g Y) = Ad_{g^{-1}}\Theta_\omega(X,Y).$$

Exercise 47.) Prove Prop. 38 in the case $G=S^1$.

Ex. Maurer-Cartan $\Rightarrow \Theta_{\omega_{triv}} \equiv 0 \Rightarrow \Theta_{\omega_{flat}} \equiv 0$

With Prop. 38: for $U \subseteq B$ s.t. $\pi^{-1}(U) \cong G \times U$ is trivial, with G abelian, the 2-form $\sigma^* \Theta_\omega \in \Gamma(\mathrm{Hom}(TU^{\wedge 2}; \mathfrak{g}))$ is indep. of the local section σ. Since E is locally trivial, the following definition thus makes sense (the nonabelian case is more involved!).

Def. The curvature two-form $K_\omega \in \Gamma(\mathrm{Hom}(TB^{\wedge 2}, \mathfrak{g}))$ is the well-defined form that coincides with $\sigma^* \Theta_\omega \in \Gamma(\mathrm{Hom}(TU^{\wedge 2}, \mathfrak{g}))$ for any local section $\sigma : U \to \pi^{-1}(U) \subseteq E$.

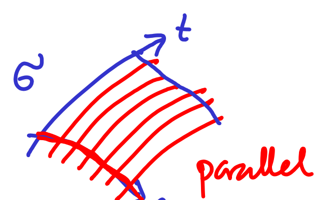
Exercise 48.)* Use Prop. 37 and Stokes' theorem to show that the parallel transport around the boundary of a square

$$u : [0,1]_s \times [0,1]_t \overset{C^\infty}{\hookrightarrow} B$$

induces a trivial monodromy $e \in G$ when $F_\omega \equiv 0$.

Hint $\int_u K_\omega \overset{\text{def.}}{=} \int_\sigma \Theta_\omega$ for a section σ, e.g. parallel along $\{t=0\}$

as well as any $\{s = cst\}$



(OK. to consider only $G = S^1$)

Proof (Prop. 38) Let U, V denote vector fields $\in \ker \omega$.

Let $X, Y \in \mathfrak{g}$. Any $T\iota_{pt} Z$, $Z \in \mathfrak{g}$, extends to the gen. v.f.

of $r_{\varphi_Z^t} : E \xrightarrow{\cong} E$, which we again call Z.

Hence $d\omega(X, Y) + [\omega(X), \omega(Y)] =$

$$\left(\substack{d \text{ \& pullback} \\ \text{commute}}\right) = d\theta(X,Y) + [\theta(X), \theta(Y)] \overset{(M-C)}{=} 0.$$

Also $d\omega(X, U) + [\omega(X), \omega(U)] =$

$$= \underbrace{d\omega(U)}_{\equiv 0} X - \underbrace{d\omega(X)}_{\equiv X} U - \omega \underbrace{[X, U]}_{\in \ker \omega} + [\omega(X), \underbrace{\omega(U)}_{\equiv 0}] \qquad \square$$

$$= 0$$

since X generates $r_{\varphi_X^t}$ which preserves $\ker \omega$.
(c.f. definition of $\mathrm{ad}_U X$)

Proof (Prop. 39.) In view of Prop. 38, it suffices to verify the formula for horizontal U, V

Tr_g-equiv. by (A2)

In this case $\Theta_\omega(U,V) = d\omega(U,V) = \omega[U,V]$

Now use the following gen. rule: $[Tf\,U, Tf\,V] = Tf[U,V]$ $\square$

Now assume $G = S^1 \Rightarrow \mathfrak{g} \cong \mathbb{R}$

Prop. 40 1.) In $G \times \mathcal{U}$, $K_\omega = d\eta$ for some $\eta \in \Gamma(\overbrace{\mathrm{Hom}(T\mathcal{U}, \mathfrak{g})}^{T^*\mathcal{U}})$

(locally exact, i.e. $K_\omega \in \Gamma(\mathrm{Hom}(TB^{\wedge 2}, \mathfrak{g}))$ is closed)

In particular: E admits a global section $\Rightarrow K_\omega = d\eta$ globally

2.) $K_\omega - K_{\omega'} = d\eta$ for some global $\eta \in \Gamma(T^*B)$.

Proof. In local triv. $G \times \overset{\subseteq B}{\mathcal{U}}$: $\omega = \theta + \overset{\in \Gamma(\mathrm{Hom}(T\mathcal{U}, \mathfrak{g}))}{A}$.

G abelian $\Rightarrow r_g^* A = A$

Change of trivialisation $(g, p) \overset{\Phi}{\mapsto} (\phi(p) \cdot g, p)$

$$\Phi^* \omega \overset{(G\text{-}E)}{=} \theta + \underbrace{d\phi}_{\text{def.: } \phi^*\theta} + A \quad \text{(see previous lecture)}$$

1.) $r_g^* A = A \Rightarrow A = \pi^* \eta \Rightarrow$

$$\sigma^* d\omega = d\sigma^*\omega = d\sigma^* A \qquad \sigma(p) = (e, p)$$

$$= d\sigma^*\pi^*\eta = d(\underbrace{\pi \circ \sigma}_{\mathrm{id}_\mathcal{U}})^*\eta$$

$$= d\eta$$

2.) $\omega - \omega' = A_\omega - A_{\omega'}$ is independent of choice of local triv.

by $(G\text{-}E) \Rightarrow A_\omega - A_{\omega'} \in \Gamma(T^*B)$ well-defined! $\square$

11. Associated vector bundles

There is a bijection:

principal G-bundles over B $\longleftrightarrow$ loc. trivial vector bundle over B with fibre V.

$G \subseteq Gl(V)$ closed — transition functions $\subseteq G$.

$$\begin{matrix} E \circlearrowleft G \\ \downarrow \\ E/G = B \end{matrix} \quad \longmapsto \quad \begin{matrix} E \times V / (x,v) \sim (x\cdot g, g\cdot v) \\ \downarrow \\ E/G = B \end{matrix}$$

bundle of "frames" $\subseteq G$, i.e. bases in G of the fibres. $\longleftarrow$ loc. triv. V-bundle

ω connection one-form $\longleftrightarrow$ covariant derivative $\nabla_X u$

sec. of v. bundle

$\in \Gamma(TB)$

$M \to \mathbb{R}$ — G-basis

$$\nabla_X \Big(\sum_i a_i \bar{e}^i\Big) =$$

$\Gamma(TM)$

framing $\leftrightarrow \bar{e}^1 \dots \bar{e}^n$

$$\sum_i da_i(X)\cdot \bar{e}^i + \underbrace{\sigma^* \omega(X)}_{\in \mathfrak{g} \subseteq \operatorname{End} V} \sum_i a_i \cdot \bar{e}^i$$

Ω_ω curvature two-form $\longleftrightarrow$ Curvature of vector-bundle

$$\in \Gamma(\operatorname{Hom}(TB^{\wedge 2}, \operatorname{End}(V)))$$

Fact: a Riemannian metric on M, i.e. a smooth choice of symm. nondegenerate $g \in \Gamma(\mathrm{Hom}(TM^{\otimes 2}, \mathbb{R}))$ induces the Levi-Civita $O(\dim M)$-connection ω^{LC} corr. to connection determined by:

- $dg(v,w)(x) = g(\nabla^{LC}_X v, w) + g(v, \nabla^{LC}_X (w))$
- $\nabla^{LC}_u v - \nabla^{LC}_v u = [u,v]$

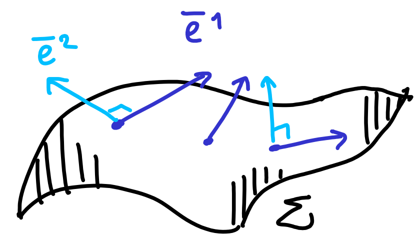
$\rightsquigarrow$ Riemannian curvature $\boxed{R_g \overset{\text{def.}}{=} K_{\omega^{LC}} \in \Gamma(\mathrm{Hom}(TM^{\wedge 2}, \mathrm{End}(TM)))}$

For an oriented surface (Σ, g) we thus have an $\boxed{\text{abelian } SO(2) = S^1\text{-frame bundle.}}$ The choice of metric g_{S^1} induces the Levi-Civita connection $\omega^{LC} \in \Gamma(\mathrm{Hom}(T\Sigma, \mathbb{R}))$.

When $\Sigma \subseteq \mathbb{R}^N$, the Euclidean metric

$$g_{\mathrm{eucl}} : T\mathbb{R}^{N\,\otimes 2} \to \mathbb{R}$$
$$((pt, X), (pt, Y)) \mapsto X \bullet Y$$

Euclidean



A field $\bar{e}^1$ of unit tangents determines a frame

restricted to $T\Sigma^{\otimes 2} \subseteq T\mathbb{R}^{N\,\otimes 2}$ has ω^{LC} determined as follows: A locally def. $SO(2)$-tangent frame $\bar{e}_1, \bar{e}_2$ is parallel along a curve $\gamma(t) \in \Sigma \longleftrightarrow \frac{d}{dt}(\bar{e}_1 \circ \gamma)(t) \perp T_{\gamma(t)}\Sigma$

10. The Gauß-Bonnet Theorem

<u>**Thm. 41**</u> (Gauß-Bonnet) $\int_{\Sigma_g} K_{\omega^{LC}} = 2\pi \overbrace{(2-2g)}^{\chi(\Sigma_g)}$.

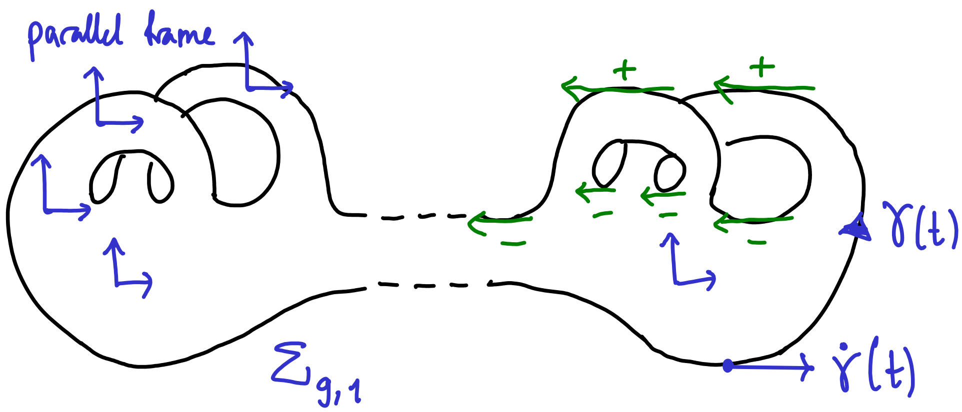
<u>Proof.</u> We can choose an <u>arbitrary</u> connection by Prop. 40 together w. Stokes' theorem

$$\int_{\Sigma_g} K_\omega \overset{(\text{Prp } 40)}{=} \int_{\Sigma_g} K_{\omega'} + d\eta = \int_{\Sigma_g} K_{\omega'} + \int_{\Sigma_g} d\eta \overset{(\text{Stokes'})}{=} \int_{\Sigma_g} K_{\omega'} + \cancel{\int_{\partial\Sigma_g} \eta}$$

since $\partial\Sigma_g = \emptyset$

Divide $\Sigma_g = \Sigma_{g,1} \cup D^2$ along $S^1 = \partial D^2$

Choose ω to coincide w. the flat connection on $S^1 \times \Sigma_{g,1}$ induced by the flat (Euclidean) frame on $\Sigma_{g,1} \hookrightarrow \mathbb{R}^2$



$\gamma : S^1 \xrightarrow{\cong} \partial\Sigma_{g,1}$

$$\Rightarrow \int_{\Sigma_{g,1}} K_\omega = \int_{\Sigma_{g,1}} K_{\omega_{triv}} = 0$$

What remains is to extend the connection to D^2.

Here we compute:

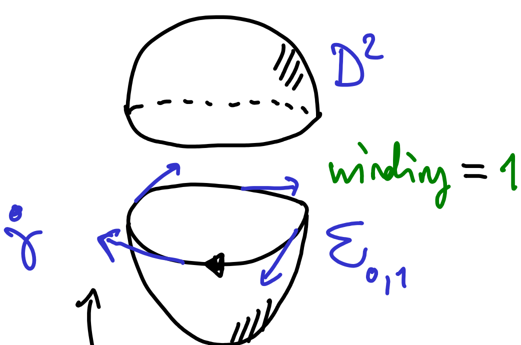
$$\int_{D^2} K_\omega \overset{\text{def}}{=} \int_{D^2} \sigma^* d\omega \overset{\text{(Ch. rule)}}{=} \int_{D^2} d\sigma^*\omega \overset{\text{(Stokes')}}{=} \int_{\partial D^2} \sigma^*\omega \overset{\text{(Prop. 37)}}{=} 2\pi(-2g+1) + 2\pi.$$

$\square$

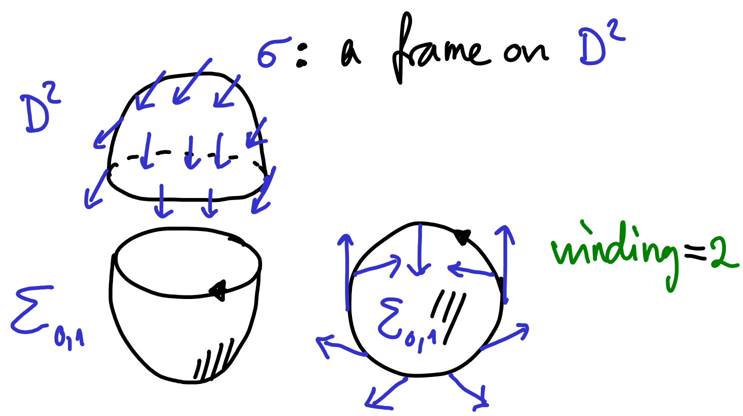
for any choice of frame σ on D^2.

<u>Exercise 49.)</u> Show the last equality by computing how many times σ must turn compared to the parallel (Euclidean frame determined by $\Sigma_{g,1}$.

<u>For S^2:</u>



$\dot\gamma$ does not extend to a frame on D^2



σ from below