georgios.dimitroglou@math.uu.se

# PhD course in Geometry & Topology

## Rough course plan

§I. Homotopy groups & fibre bundles

§II. (Morse) homology of smooth manifolds

§III. Knot theory

§IV. Gauge theory

## §0. Introduction

Topology studies e.g. properties invariant under

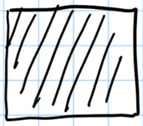
- homotopy equivalence (see below)
- homeomorphism ($\varphi$ & $\varphi^{-1}$ cont)
- diffeomorphism ($\varphi$ & $\varphi^{-1}$ smooth)

Geometry is typically more rigid, studies e.g. properties invariant under

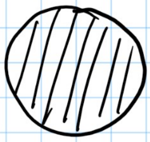
- isometry

## Ex

(i)  $\simeq_{ht}$ • point — homotopy equivalence

(ii)  $\cong_{C^0}$  — homeomorphic domains

$\cancel{\cong_{C^\infty}}$

(iii)  $\cong_{C^\infty}$  — diffeomorphic domains

(iv) $D \subseteq \mathbb{R}^n$ connected open domain

Any function $\varphi : D \to \mathbb{R}^n$ that preserves distances is clearly continuous & injective.

In fact:

**Exercise 1*** $\varphi$ is the restriction of an $\underbrace{\text{affine Euclidean isometry}}_{\text{"rigid transformation"}}$ $\mathbb{R}^n \to \mathbb{R}^n$

Hint: Consider the case $\varphi(0) = 0$ and show that $\varphi$ preserves the std. inner product

A **topology** on a set $X$ is a collection of subsets $\Omega \subseteq \mathcal{P}(X)$ called "open" subject to: • $\emptyset, X \in \Omega$, $\Omega$ closed under • arbitrary unions
• finite intersections

$\varphi : (X, \Omega_X) \to (Y, \Omega_Y)$ **continuous** if $\boxed{U \in \Omega_Y \Rightarrow \varphi^{-1}(U) \in \Omega_X}$

We will here only consider topologies induced by a metric:

**Def** $d : X \times X \to \mathbb{R}_{\geq 0}$ is a **metric** if

(M1) $d(x,y) = 0 \Leftrightarrow x = y$ (non-degenerate)

(M2) $d(x,y) = d(y,x)$ (symmetric)

(M3) $d(x,z) \leq d(x,y) + d(y,z)$ (triang. ineq)

$B_r(x) := \{ y \in X ; d(x,y) < r \}$ open ball w.r.t the metric $d$

**Basic facts**

- $\Omega_d := \left\{ U \subseteq X ; U = \bigcup_{\substack{x \in X \\ r_x \geq 0}} B_{r_x}(x) \right\}$ is a topology
- $\Omega_d = \Omega_{d'} \Leftrightarrow d : \underbrace{X \times X}_{\text{top induced by } d'} \to \mathbb{R}$ cont. w.r.t $d'$ **and** $d' : \underbrace{X \times X}_{\text{top induced by } d} \to \mathbb{R}$ cont. w.r.t $d$

## Nagata-Smirnov metrisation theorem ('50)

$(X, \Omega)$ topological space

$\Omega$ is induced by a metric $\Leftrightarrow$

- Regular 
- Hausdorff 
- basis consisting of a countable union of locally finite collections of subsets

fails for the Zariski topology

**From now on:** we consider only topologies induced by metrics

e.g. subsets of $\mathbb{R}^n$ with the subspace topology

## Important categories

**Top:** Category of topological spaces $(\subseteq \text{Set})$

**Ob:** $(X, \Omega)$

**Mor:** $\text{Hom}((X, \Omega_X), (Y, \Omega_Y)) = \underbrace{C((X, \Omega_X), (Y, \Omega_Y))}_{\text{continuous maps}} = C(X, Y)$

Isomorphisms are cont maps w. cont. inverse, i.e. **homeomorphisms**

$\underline{\mathrm{Top}_*}$ Category of based topological spaces

$\underline{\mathrm{Ob}}$: $(X, \Omega, pt)$, $pt \in X$,

$\underline{\mathrm{Mor}}$: $\mathrm{Hom}((X, pt_X), (Y, pt_Y)) = C((X, pt_X), (Y, pt_Y))$

$= \{ \varphi \in C(X,Y) \text{ s.t. } \varphi(pt_X) = pt_Y \}$

## Homotopy

Captures the more coarse behaviour of $C(X,Y)$

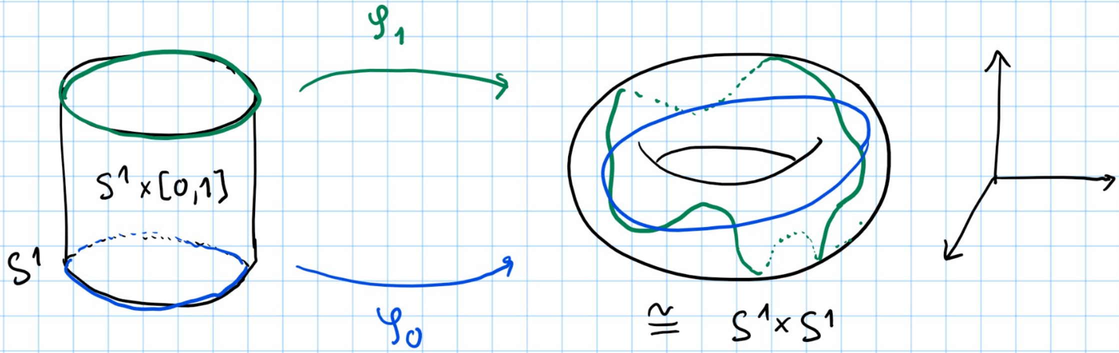
**Def** Two maps $\varphi_i \in C(X,Y)$ are *homotopic*, written $\varphi_0 \sim \varphi_1$,

if there is $\Phi : X \times [0,1] \to Y \in C(\overbrace{X \times \underbrace{[0,1]}_{\text{std top}}}^{\text{prod top.}}, Y)$

s.t. $\Phi|_{X \times \{i\}} = \varphi_i$, $i = 0,1$,

$$\boxed{[X,Y] := C(X,Y)/\sim}$$

**Ex**




**Exercise 2** Show that

(i) homotopy is an equivalence relation

(ii) composition of homotopic maps are homotopic

(iii) the induced relation $\circ : [Y,X] \times [Y,Z] \to [X,Z]$ is associative

In view of this, the equiv classes themselves form the morphism spaces of a category: $\underline{h\mathrm{Top}}$

## hTop: the (naive) homotopy category

**Ob** $(X, \Omega)$ same as usual

**Mor**: $[X,Y]$ identity: $[\mathrm{id}_X] \in [X,X]$

Isomorphisms: homotopy equivalences i.e. $\varphi \in C(X,Y)$ s.t. $\exists \psi \in C(Y,X)$ $\psi\circ\varphi \sim \mathrm{id}_X$, $\psi \circ \varphi \sim \mathrm{id}_Y$

Similarly, one defines the based homotopy category

**$hTop_*$** **Ob** $(X, pt)$

**Mor** $[(X,pt_X),(Y,pt_Y)] := C((X,pt_X),(Y,pt_Y))/$ homotopy that fixes $pt_X$, i.e $\Phi(pt_X, t) = pt_Y$

**Ex** (i) $\mathbb{R}^n \simeq \{0\} \simeq B^n \simeq D^n = \overline{B^n}$ (or generally: starshaped domains)

$\mathrm{id}_{\mathbb{R}^n} \sim (\bar{x} \mapsto 0)$ Homotopy: $\bar{x}\cdot t$ $\bar{x} \in \mathbb{R}^n$, $t \in [0,1]$

(ii) $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1} \simeq S^{n-1} \times \mathbb{R}$

$\mathrm{id}_{\mathbb{R}^n \setminus \{0\}} \sim \left(\bar{x} \mapsto \frac{\bar{x}}{\|\bar{x}\|}\right)$ Homotopy: $(1-t)\bar{x} - t\frac{\bar{x}}{\|\bar{x}\|}$

In conclusion:

- $[X, \{pt\}] = \{cst\}$ is a singleton
- $[\{pt\}, X] = \{\text{path components in } X\} =: \boxed{\pi_0(X)}$

$pt \mapsto x$

Homotopy = path

$pt \mapsto y$

Thus: $[X, \mathbb{R}^n] \underset{bij}{\cong} [X,\{0\}] = \{cst\}$ etc

**Exercise 3** Use Gram-Schmidt to show that

$$\underset{\subseteq \mathbb{R}^{n\cdot n}}{Gl_n(\mathbb{R})} \simeq O(n) \quad \& \quad Gl_n(\mathbb{C}) \simeq U(n)$$

In particular: $Gl_2(\mathbb{R}) \simeq S^1 \amalg S^1$, $Gl_1(\mathbb{C}) \simeq S^1$

# Smooth manifolds  A class of well-behaved spaces & maps
$C^\infty$

$\underline{\mathrm{Man}^0} \subseteq \underline{\mathrm{Top}}$ full subcategory of metrisable spaces that are locally homeomorphic to $\mathbb{R}^n$: topological manifolds

Not quite good enough...

$\underline{\mathrm{Man}^\infty}$ Smooth manifolds $\boxed{1^{st} \text{ def:}}$

part of the data  not part of the data

<u>Ob</u>: Subsets $M \subseteq \mathbb{R}^n$ that <u>locally</u> is the vanishing locus of a smooth funct.
<u>More precisely</u>: each $x \in M$ admits an open nbhd $\mathcal{U} \subseteq \mathbb{R}^n$ in which $\mathcal{U} \cap M = f^{-1}(0)$ for $f: \mathcal{U} \to \mathbb{R}^m$ a $C^\infty$ function with $0 \in \mathbb{R}^m$ a $\boxed{\text{reguler value}}$ $Df|_{f^{-1}(0)}$ full rank

<u>Mor</u>: $\mathrm{Hom}(M,N) = C^\infty(M,N) :=$

$$\left\{ \varphi \in C(M,N) \;\middle|\; \begin{array}{l} \text{for all } x \in M \text{ there is a nbhd } \mathcal{U} \subseteq \mathbb{R}^{n_M} \\ \text{and a smooth } \Phi: \mathcal{U} \to \mathbb{R}^{n_N} \text{ s.t. } \Phi|_{\mathcal{U}\cap M} = \varphi|_{\mathcal{U}\cap M} \end{array} \right\}$$

isomorphisms are called diffeomorphisms: $\varphi$ & $\varphi^{-1}$ exist and are $C^\infty$

$\boxed{2^{nd} \text{ def:}}$

<u>Ob</u>: $M$ $2^{nd}$ countable metrisable topological space with the <u>choice</u> of an open cover $\{\mathcal{U}_i\}$ together with homeomorphisms $\varphi_i : \mathcal{U}_i \xrightarrow{\cong} \mathbb{R}^n$
s.t. $\varphi_j \circ \varphi_i^{-1}$ are smooth for all $i,j$ where defined  $\varphi_i(\mathcal{U}_j) \subseteq \mathbb{R}^n$

<u>Mor</u> $\mathrm{Hom}(M,N) = C^\infty(M,N) :=$

$$\left\{ \varphi \in C(M,N) \;\middle|\; \begin{array}{l} \varphi_j^N \circ \varphi \circ (\varphi_i^M)^{-1} \text{ is } C^\infty \text{ for all } i,j \\ \text{where defined} \end{array} \right\}$$

<u>Rmk</u>

$$1^{st} \text{ def.} \underset{\text{Whitney embedding thm}}{\overset{\text{implicit function thm}}{\rightleftarrows}} 2^{nd} \text{ def.}$$