 georgios.dimitroglou@math.uu.se

Ex Simple examples of manifolds

- open subsets of $\mathbb{R}^N$, e.g. $Gl_n(\mathbb{R})$, $Gl_n(\mathbb{C})$
- $S^n = \{\bar{x} \in \mathbb{R}^{n+1} \mid \underbrace{\|\bar{x}\|^2}_{\text{smooth fcn}} = 1\}$ ← regular value — a vanishing locus of a globally defined function
- $S^n \times S^m = \{(\bar{x},\bar{y}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \mid \|\bar{x}\| = \|\bar{y}\| = 1\}$

The classification question (up to homeo/diffeo) is more tractable for manifolds than for general topological spaces.

E.g.: a compact manifold that can be covered by two coordinate charts $M \overset{\text{open}}{\supseteq} U \longrightarrow \mathbb{R}^n$ is homeomorphic to S^n.

(Key ingredient: Jordan-Schoenflies thm: $S^{n-1} \xhookrightarrow[\text{tame}]{} \mathbb{R}^n$ bounds a ball)

⚠ [Milnor '57] showed that ∃ precisely 28 different diffeomorphism classes of smooth manifolds that are homeomorphic to S^7.

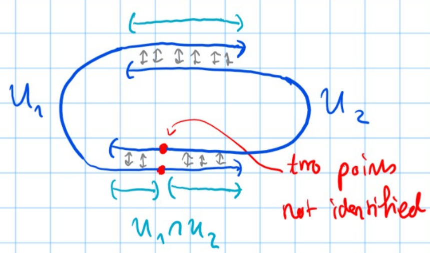
Exercise 4* a) Show that any compact 1-dim. manifold which can be covered by two charts is homeomorphic to S^1

b) Show that any connected & compact 1-dim. manifold can be covered by two coordinate charts,

Hint: The Hausdorff property (uniqueness of limits) is crucial

non-Hausdorff "circle":

$U_1 \cap U_2$ has three components



The structure of manifolds will be further explored in §II

§I Homotopy groups

<u>1st goal</u>: $[S^k, S^n] = [(S^k,*),(S^n,*)] = \begin{cases} \mathbb{Z}, & k=n \\ 0, & k<n \end{cases}$

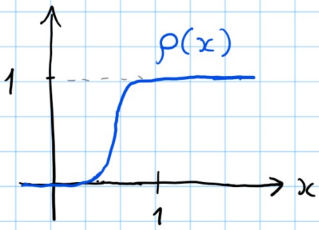
The homotopy groups

$(X,*) \in \mathcal{T}op_*$ based topological space

$$\boxed{\pi_k(X,*) := [(S^k,N),(X,*)]}$$

$N = (0,0,\ldots,0,1) \in S^k$ "North pole", $k \geq 0$

The k-th homotopy group



<u>Recall</u>: • $S^k \setminus \{pt\} \cong \mathbb{R}^k$

• $\exists$ smooth $\psi_t(\bar{x}) = (1-t)\cdot\bar{x} + t\rho(\|\bar{x}\|)\cdot\bar{x}$ $\quad t \in [0,1]$

$\psi_0 = id$, $\psi_1(B^k_\varepsilon) = \{0\}$

<u>Conclusion</u>: • Any element $\gamma : (S^k,N) \to (X,*)$ is homotopic to $\gamma \circ \psi_1 : (S^k,N) \to (X,*)$ which is equal to $*$ in a neighbourhood of N

• The same can also be assumed for any homotopy of maps $(S^k,N) \to (X,*)$

• Hence, instead of htpy classes $[(S^k,N),(X,*)]$ we can study htpy classes of maps $\mathbb{R}^k \to X$ which are $\equiv *$ outside of some compact subset.

The group structure

$[\gamma_0],[\gamma_1] \in \pi_k(X,*)$, $k \geq 1$. Represent classes by $\tilde{\gamma}_i : \mathbb{R}^k \to X$

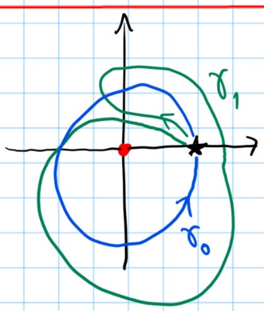
$\tilde{\gamma}_0 = \gamma_0(x_1 + A, x_2, \ldots, x_k) \quad = * \quad x_1 \geq 0$

$\tilde{\gamma}_1 = \gamma_1(x_1 - A, x_2, \ldots, x_k) \quad = * \quad x_1 \leq 0$

when $A \gg 0$

$$\boxed{[\gamma_0]\cdot[\gamma_1] := \left[\bar{x} \longmapsto \begin{cases} \tilde{\gamma}_0(\bar{x}) & x_1 \leq 0 \\ \tilde{\gamma}_1(\bar{x}) & x_1 \geq 0 \end{cases} \right] \in \pi_k(X,*)}$$

$(\mathbb{R}^2 \setminus \{0\}, \bigstar = (1,0))$



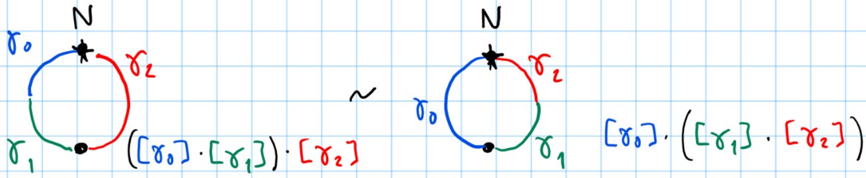
$[\gamma_0]\cdot[\gamma_1] = [cst_\bigstar : S^1 \to \{\bigstar\} \subseteq \mathbb{R}^2]$

Prop The above operation makes $\pi_k(X,*)$, $k \geq 1$, into a group which is abelian when $k \geq 2$,

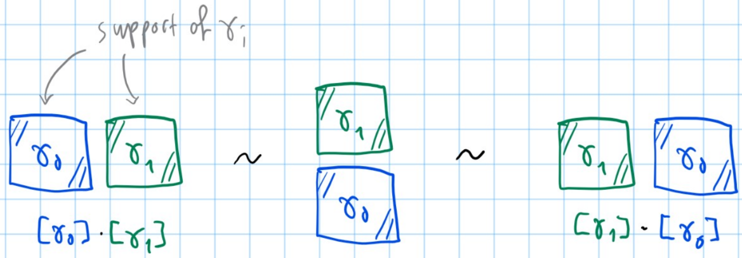
- unit $= [cst_*: S^k \to \{*\} \subseteq X]$
- $[\gamma]^{-1} = [\gamma(-x_1, x_2, \dots, x_k)]$

$\pi_1(X,*)$ is called the fundamental group

Proof Associativity:



Abelian for $k \geq 2$:



Inverse property:

$$\bar{x} \longmapsto \begin{cases} \gamma(-x_1+t, x_2, \dots, x_k), & x_1 \leq 0 \\ \gamma(x_1+t, x_2, \dots, x_k), & x_1 \geq 0 \end{cases} \quad (\equiv * \text{ for } t \gg 0)$$

(depends continuously on t)

$\square$

Prop Post-composition with $f: (X, pt_X) \to (Y, pt_Y)$ induces a homomorphism $f_*: \pi_k(X, pt_X) \to (Y, pt_Y)$ of homotopy groups, that only depends on $[f] \in [(X, pt_X), (Y, pt_Y)]$

Thm $(\pi_k(S^n,*),\cdot) \cong \begin{cases} 0, & k<n \\ (\mathbb{Z},+), & k=n \end{cases}$ when $n\geq 1$

Moreover: $\pi_k(S^n,*) \xrightarrow{} [S^k, S^n]$ is a bijection for all $n \geq 1$

forgetting the basepoint

Proof Recall that any $[\gamma] \in \pi_k(X,*)$ can be represented by

$\gamma: \mathbb{R}^k \to X \quad \gamma \equiv *$ near ∞

After a further rescaling: γ can be identified with

$$\gamma: [0,1]^k \to X \qquad \gamma\big|_{\partial [0,1]^k} \equiv *$$

We may replace $(S^n,*)$ with $(\mathbb{R}^{n+1}\setminus\{0\},*) \supseteq (S^n,*)$

(Recall the homotopy $\mathbb{R}^{n+1}\setminus\{0\} \simeq S^n$ given by $\bar{x} \mapsto (1-t)\bar{x} + t\frac{\bar{x}}{\|\bar{x}\|}$)

For $\gamma: [0,1]^k \to \mathbb{R}^{n+1}\setminus\{0\}$ fixed we can find $M \gg 0$ s.t.

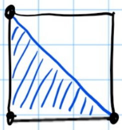
the map $\tilde{\gamma}: [0,1]^k \to \mathbb{R}^{n+1}$ defined by:

- $\tilde{\gamma} = \gamma$ on $\underbrace{\{0, 1/M, 2/M, \dots, (M-1)/M, 1\}^k}_{\text{vertices of subdivision of } [0,1]^k \text{ into } M^k \text{ sub-cubes}} \subseteq [0,1]^k$

- $\tilde{\gamma}$ is affine-linear on each k-simplex arising in a subdivision of the cubes $[0,1/M]^k + (i_1/M, \dots, i_k/M) \quad i_j \in \{0,1,\dots,M\}$ into affine-linear simplices (triangulation)

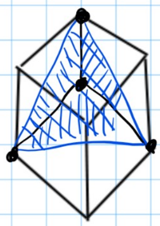
The k-dimensional simplex (Digression)

$\Delta_k := \{ \bar{x} \in [0,1]^k \mid x_1 + \dots + x_k \leq 1 \}$ = convex hull of the k+1 vertices $(0,\dots,0), (1,0,\dots,0), \dots, (0,\dots,0,1)$



$\Delta_2 \subseteq [0,1]^2$

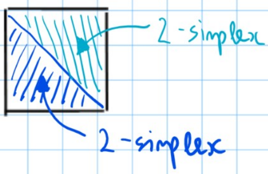
vol = 1/2



$\Delta_3 \subseteq [0,1]^3$

$\mathrm{vol} = 1/2^{3-1}$

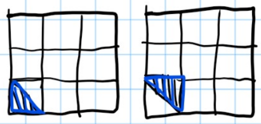
Each cube can be subdivided into affine-linear simplices (triangulated) e.g. in the following manner:



1) connect all 2^k vertices by a sequence of $2^k - 1$ distinct edges
2) mark the 0:th, 2:nd, 4:th, 6:th, ... vertices in the path ($2^k/2 = 2^{k-1}$ number of marked vertices)
3) take the simplex spanned by the 2^{k-1} marked vertices & their k adjacent vertices (of distance one)

(Since there is no loop consisting of an odd nr. of edges, the latter k vertices are all unmarked)

If $M \gg 0$ then $\tilde{\gamma}$ is a good approximation of γ



$k=2$
$M=3$

In particular:

- $\tilde{\gamma}$ takes values in $\mathbb{R}^{n+1} \setminus \{0\}$
- $(1-t)\gamma + t\tilde{\gamma}$ is a homotopy of maps $(S^k, N) \to (\mathbb{R}^{n+1} \setminus \{0\}, *)$

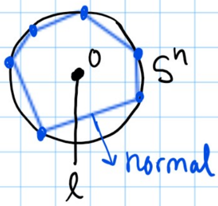
$\tilde{\gamma}$ parametrises a finite nr of affine-linear (possibly degenerate) simplices in $\mathbb{R}^{n+1} \setminus \{0\}$ after a perturbation, we may assume that these affine-linear simplices all are in general position w.r.t. the line

$$\ell := \{(0,\dots,0,t) \mid t<0\} \subseteq \mathbb{R}^{n+1} \setminus \{0\}$$

I.e. all simplices that meet ℓ are

- non-degenerate & of dimension $\geq n$
- have normals $\notin \ell^{\perp}$
- meet ℓ in a single point in the interior if $\dim = n$

general pos:



Rmk General cont. maps are not well behaved (c.f. space filling curves)
Piecewise linearity can be replaced by smoothness & we can use Sard's theorem to achieve general position.

Case $k<n$: General position w.r.t $\ell \iff \tilde{\gamma}$ has image disjoint from ℓ

Since $S^n \setminus \{(0,\dots,0,-1)\} \cong \mathbb{R}^n$ and $\mathbb{R}^n \simeq \{*\}$

we conclude that $\pi_k(S^n, *) = 0$ when $k<n$ as sought.