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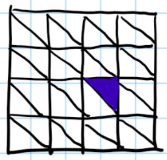
We continue the calculation $[S^k, S^n] \cong \pi_k(S^n) = \begin{cases} 0 & k<n \\ \mathbb{Z} & k=n \end{cases}$ from last time

<u>Recall</u>: We have replaced $\gamma \in \pi_k(S^n,*)$ with a simplicial $\tilde{\gamma}:[0,1]^k \to \mathbb{R}^{n+1}\setminus\{0\}$

$\tilde{\gamma}:$ 

$[0,1]^k$

$\xrightarrow{\boxed{\text{affine linear on each simplex}}} \mathbb{R}^{n+1}\setminus\{0\} \supseteq S^n$ & in general position with respect to the line $\ell = \{(0,\ldots,0)\}\times\mathbb{R}_{<0}$

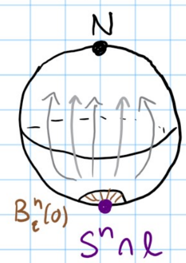
<u>Case $k=n$</u>: After post-composing $\tilde{\gamma}/\|\tilde{\gamma}\| \sim \tilde{\gamma}$ with a map

$$(S^n,N)\to(S^n,N) \sim \mathrm{id}_{(S^n,N)}$$

that sends all except some small nbhd of $S^n\cap\ell$ to N, we get that

$\tilde{\gamma}/\|\tilde{\gamma}\| \sim \gamma_1\cdot\ldots\cdot\gamma_K$ where

$\gamma_i: \mathbb{R}^n \to S^n$ satisfies



- $\gamma_i^{-1}(S^n\cap\ell) = \{0\}\in\mathbb{R}^n$
- $\gamma_i|_{B^n_\varepsilon(0)}: B^n_\varepsilon(0) \to S^n\setminus\{N\}\cong\mathbb{R}^n$ satisfies $D\gamma_i(0)\in GL_n(\mathbb{R})$

Further homotopies $\sim \tilde{\gamma}_i|_{\mathbb{R}^n} = D\gamma_i(0): \mathbb{R}^n \to S^n\setminus\{N\}\cong\mathbb{R}^n$

$\sim \tilde{\tilde{\gamma}}_i|_{\mathbb{R}^n} = \underbrace{(\pm x_1, x_2, \ldots, x_n)}_{[\mathrm{id}_{S^n}]^{\pm 1}}: \mathbb{R}^n \to S^n\setminus\{N\}\cong\mathbb{R}^n$ $(*)$

(See Lecture 02)

<u>Exercise 5</u> Give the details of $(*)$

We conclude that any $\gamma\in\pi_n(S^n,N)$ can be written as

$\gamma \sim \underset{\tilde{\tilde{\gamma}}_1}{[\mathrm{id}_{S^n}]^{\pm1}}\cdot\underset{\cdots}{\ldots}\cdot[\mathrm{id}_{S^n}]^{\pm1}$, i.e. $\pi_n(S^n,N)$ is generated by $[\mathrm{id}_{S^n}]$.

($\cong \mathbb{Z}/m\mathbb{Z}$ but what is the order of $[\mathrm{id}_{S^n}]$?)

<u>Exercise 6</u> below shows that $\pi_n(S^n,N)\cong\mathbb{Z}$

<u>Finally: $\pi_k(S^n,*)\to[S^k,S^n]$ is a bijection</u> (for all $k\geq 0$)

<u>Surjective</u>: since $SO(n+1) \circlearrowright S^n$ acts smoothly & <u>transitively</u>

<u>Injective</u>: If $\gamma_t : S^k \to S^n$ is a homotopy (not neccesarily preserving basepoints) between $\gamma_0, \gamma_1 : (S^k, N) \to (S^n, N)$, we can make it basepoint preserving after

<u>$n=1$</u>: A family of rotations $\in SO(2) \cong S^1$ (cont. depending on $\gamma_t(N)$)

<u>$n>1$</u>: $\gamma_t(N) : [0,1] \to S^n$ can be assumed to be piecewise linear by the above, and hence misses a generic point $0 \in \mathbb{R}^n = S^n \setminus \{N\}$

<u>Finally</u>: post-compose γ_t with a map $(S^n, N) \to (S^n, N) \sim id_{(S^n,N)}$ which sends everything outside $B_2^n \subseteq \mathbb{R}^n \subseteq S^n \setminus \{N\}$ to N. $\square$

Exercise 6

Show that the map

$$\begin{aligned} C^{p.w.\infty}(S^n, \mathbb{R}^{n+1} \setminus \{0\}) & \\ \gamma \longmapsto \oint_\gamma \vec{F} \cdot \vec{n} \, dS =: \boxed{\mathrm{Wind}(\gamma)} & \end{aligned}$$

the <u>winding number</u> of γ

For $\vec{F}(x_1, \dots, x_{n+1}) = \dfrac{1}{\mathrm{Area}(S^n) \cdot \|\vec{x}\|^{n+1}} \cdot (x_1, \dots, x_{n+1})$ a vector field on $\mathbb{R}^{n+1} \setminus \{0\}$

(expressed as a differential n-form: $\dfrac{1}{\mathrm{Area}(S^n) \cdot \|x\|^{n+1}} \cdot \iota_{\sum_i x_i \partial_{x_i}} dVol_{\mathbb{R}^{n+1}}$)

induces a group homomorphism $\pi_n(S^n, *) \xrightarrow{\cong} \mathbb{Z}$

$$[id_{S^n}] \longmapsto 1$$

<u>Immediate conclusion</u>:

- since $\pi_n(S^n, N)$ is generated by $[id_{S^n}]$, the above is actually an <u>isomorphism</u> of groups $\pi_n(S^n, N) \xrightarrow{\cong} \mathbb{Z}$
- the winding number is always <u>integer valued</u>

Useful terminology: X <u>contractible</u> if $X \simeq_{hTop} \{pt\}$

X <u>k-connected</u> if $\pi_k(X) = \{0\}$ for $k = 0, 1, 2, \dots, k$.

1-conn. is also called <u>simply connected</u>

<u>Thm</u> [Whitehead] For well-behaved spaces (CW-complexes, e.g. manifolds)

$$X \text{ contractible} \iff \pi_k(X) = 0 \text{ for all } k = 0, 1, 2, \dots$$

Dependence on basepoint

Exercise 7 For any path $\eta : [0,1] \to X$, construct a "natural" isomorphism $\varphi_\eta : \pi_k(X, \eta(0)) \xrightarrow{\cong} \pi_k(X, \eta(1))$ which

(i) is invariant under homotopy of η rel. endpoints

(ii) $\varphi_\delta \circ \varphi_\eta : \pi_k(X, \eta(0)) \longrightarrow \pi_k(X, \delta(1))$

$= \varphi_{\eta \star \delta}$ concatenation of paths

(iii) $\varphi_\gamma \in \mathrm{Aut}(\pi_1(X, \gamma(0))$ for $\gamma \in \pi_1(X, \gamma(0))$

is conjugation with γ $\quad x \mapsto \gamma x \gamma^{-1}$ (an "inner automorphism")

The above implies that the isomorphism class of $\pi_k(X,*)$ only depends on the path component of $* \in X$. ⚠ The isomorphism is not canonical (dep^s on choice of path)

i.e. The functor $\underline{hTop_*} \longrightarrow \underline{Gp}$

$\underline{Ob}$: $\quad (X,*) \longmapsto \pi_1(X,*)$

$\underline{Mor}$: $[f] \in [(X, pt_X), (Y, pt_Y)] \longmapsto \left\{ \begin{array}{l} f_* : \pi_1(X, pt_X) \to \pi_1(Y, pt_Y) \\ \quad [\gamma] \longmapsto [f \circ \gamma] \end{array} \right\}$

Does not descend to a functor $\underline{hTop} \to \underline{Gp}$ $\quad X \longmapsto \pi_1(X, \underset{\substack{\uparrow \\ \text{non-natural}}}{*})$

However, after abelinising, we can forget the basepoint:

abelian groups

$$\underline{hTop} \longrightarrow \underline{AbGp}$$

the abelinisation

$\underline{Ob}$ $\quad X \mapsto \bigoplus_{* \in \pi_0(X)} \pi_1(X,*) / [\pi_1(X,*), \pi_1(X,*)] =: \boxed{H_1(X)}$ the 1^{st} homology group

$\underline{Mor}$ $\quad [X,Y] \to \mathrm{Hom}(H_1(X), H_1(Y))$

$\qquad [f] \mapsto ([\gamma] \mapsto [f \circ \gamma])$

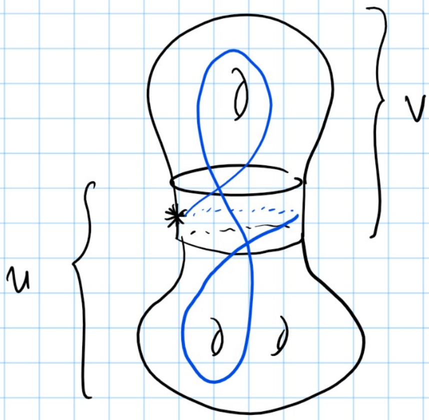
the choice of $*$ is not natural, but $H_1(X)$ for different choices of $*$ are canonically isomorphic.

Seifert – van Kampen's theorem

A powerful tool for computing $\pi_1(X)$ by decomposition

Thm If $X = U \cup V$, where U, V, and $U \cap V$ are open $\subseteq X$ & path-connected, then the canonical inclusions $\iota_{A \subset B} : A \hookrightarrow B$ induce a pushout diagram of groups for any $* \in U \cap V$

$$\begin{tikzcd}
\pi_1(U \cap V, *) \arrow[r, "(\iota_{U\cap V \subset U})_*"] \arrow[d, "(\iota_{U \cap V \subset V})_*"'] \arrow[dr, phantom, "\cong"] & \pi_1(U, *) \arrow[d, "(\iota_{U \subset X})_*"] \\
\pi_1(V, *) \arrow[r, "(\iota_{V \subset X})_*"'] & \pi_1(X, *)
\end{tikzcd}$$



is a pushout diagram, in particular $\pi_1(X,*) \cong \pi_1(U,*) \underset{\pi_1(U\cap V,*)}{*} \pi_1(V,*)$

Fact The push-out is a universal construction that is isomorphic to the "amalgamated product", i.e.

$$\begin{tikzcd}
H \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"'] & G_1 \arrow[d, "\beta_1"] \\
G_2 \arrow[r, "\beta_2"'] & G_1 \underset{\alpha_1,\alpha_2}{*} G_2
\end{tikzcd}$$

where $G_1 \underset{(\alpha_1,\alpha_2)}{*} G_2 = G_1 \overset{\text{free product}}{*} G_2 / N$ $\quad$ $N < G_1 * G_2$ the smallest normal that contains $\alpha_1(h) \cdot \alpha_2(h^{-1})$

$G_i = \langle \underbrace{g_{i,1}, g_{i,2}, \dots}_{\text{generators}} \mid \underbrace{r_{i,1}, r_{i,2}, \dots}_{\text{relations}} \rangle = \underbrace{\langle g_{i,1}, g_{i,2}, \dots \rangle}_{\text{free gp gen. by } g_{i,j}} / \text{normal subgp gen by } r_{i,j}$

e.g.

$$\begin{tikzcd}
h \arrow[r, mapsto] \arrow[d, mapsto] & \alpha_1(h) \arrow[d, mapsto] \\
\alpha_2(h) \arrow[r, mapsto] & {[\alpha_2(h)] = [\alpha_1(h)]}
\end{tikzcd}$$

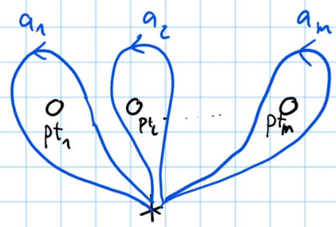
Ex $\pi_1(U\cap V,*) = \langle a_1,\dots,a_m\rangle$

$\pi_1(V,*) = \langle b_1,\dots,b_k\rangle \qquad \pi_1(U,*) = \langle c_1,\dots,c_p\rangle$ } free groups

Then $\pi_1(X,*) = \langle \underbrace{b_1,\dots,b_k,c_1,\dots,c_p}_{\text{generators}} \mid \underbrace{(\iota_{U\cap V\subset U})_*(a_i)\cdot(\iota_{U\cap V\subset V})_*(a_i^{-1})}_{\text{relations}} \rangle$

i.e the free group on $k+p$ generators quotiented by the normal subgroup generated by the relations.

Ex $\pi_1(\mathbb{R}^2\setminus\{pt_1,\dots,pt_m\},*) = \underbrace{\langle a_1,\dots,a_m\rangle}_{\text{free group on } m \text{ generators}}$



Pf $\mathbb{R}^2\setminus\{pt\}\simeq S^1$, so statement true for $m=1$

By induction: $\mathbb{R}^2\setminus\{pt_1,\dots,pt_m\} \cong \overbrace{\{x\le\tfrac12 \mid (x,y)\ne(-(m+1),0),(-m,0),\dots,(-1,0)\}}^{U\,\cong\,\mathbb{R}^2\setminus\{pt_1,\dots,pt_{m-1}\}}$

$$\cup$$

$$\underbrace{\{x\ge-\tfrac12 \mid (x,y)\ne(1,0)\}}_{V\,\cong\,\mathbb{R}^2\setminus\{pt_m\}}$$

$U\cap V = \{x\in[-\tfrac12,\tfrac12]\}\simeq\{0\}$

and the statement now follows by Seifert van-Kampen's thm. $\square$

Thm For $\Gamma\subseteq\mathbb{R}^m$ a connected graph $\pi_1(\Gamma)$ is a free group

Further $\pi_1(\Gamma)=0 \Leftrightarrow \Gamma$ is a tree.