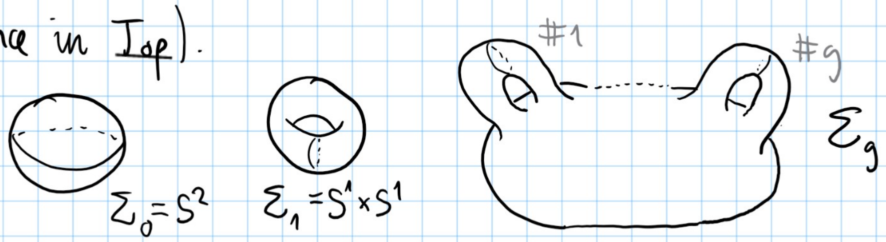
Exercise 8 Show that any finite connected graph is homotopy equivalent (isom. in $\underline{hTop}$) to a graph with <u>only one vertex</u>, and use this to compute π_1.

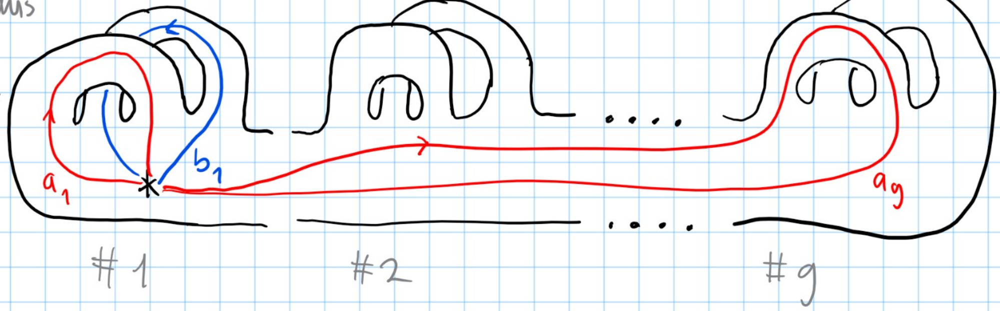
Later we will discuss the classification of surfaces; the following exercise shows that Σ_g, the "closed surfaces of genus $g \geq 0$", are pairwise non-isomorphic in $\underline{hTop}$ (& hence in $\underline{Top}$).



Exercise 9 a) Use Seifert–van Kampen to compute π_1 of

assume this

$\Sigma_{g,1} := \Sigma_g \setminus \text{Ball} \cong$



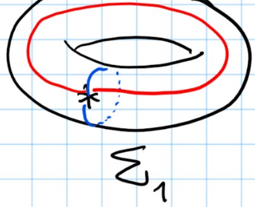
$(\Sigma_{0,1} \cong D^2)$

I.e the <u>surface of genus $g \geq 0$ and one boundary component</u>

Note that the boundary is indeed connected $\partial \Sigma_{g,1} \cong S^1$

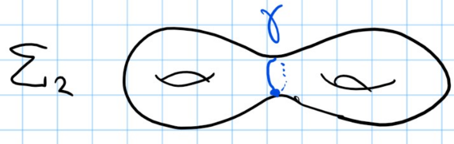
$$\boxed{\Sigma_g := \Sigma_{g,1} \cup_{\partial \Sigma_{g,1}} D^2}$$

<u>The closed surface of genus $g \geq 0$</u>



b) Show that $\pi_1(\Sigma_g) \cong \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid \underbrace{a_1 b_1 a_1^{-1} b_1^{-1}}_{\text{Commutator } [a_1, b_1]} \cdots \underbrace{a_g b_g a_g^{-1} b_g^{-1}}_{[a_g, b_g]} \rangle$

c) Show that $H_1(\Sigma_g) \cong \mathbb{Z}^{2g}$ ← hence g distinguishes Σ_g



$[\gamma] = 0 \in H_1(\Sigma_2)$

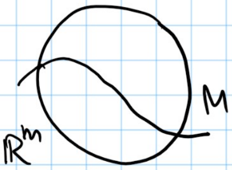
but $[\gamma] \neq 0 \in \pi_1(\Sigma_2)$

The Tangent Bundle

If $M \subseteq \mathbb{R}^m$ is a smooth manifold, then

$$\underline{TM} := \{(x, \bar{v}) \in M \times \mathbb{R}^m \mid D_x \varphi(\bar{v}) = 0 \text{ for all } \varphi \in C^\infty(\mathbb{R}^m, \mathbb{R}) \text{ s.t. } \varphi|_M \text{ constant}\}$$

is a smooth manifold called the <u>tangent bundle of M</u>.



- $M \cap \mathcal{U} = f^{-1}(0) \qquad f : \mathcal{U} \to \mathbb{R}^{m-n}, \quad Df|_{M \cap \mathcal{U}} \text{ surj} \qquad \mathcal{U} \subseteq \mathbb{R}^m$

$\Rightarrow \quad TM \cap (\mathcal{U} \times \mathbb{R}^m) = (f, Df)^{-1}(0,0) \qquad (x, \bar{v}) \mapsto (f(x), D_x f(\bar{v}))$ has $0 \in \mathbb{R}^{2(m-n)}$ as a regular value

- $p_{TM} : TM \twoheadrightarrow M$ is a smooth surjection with $p_{TM}^{-1}(x) \subseteq \{x\} \times \mathbb{R}^m$ a <u>linear subspace of dimension $n = \dim M$</u>

$$\begin{tikzcd} TM \arrow[r, two heads] \arrow[d, phantom, "\cap"] & M \\ M \times \mathbb{R}^m \arrow[ur, "\text{proj}"'] \arrow[u, phantom, "\circlearrowleft", shift left=3] & \end{tikzcd}$$

(i.e $\psi = \varphi^{-1}$ inverse of a chart)

- If $\psi : \mathbb{R}^n \hookrightarrow M$ is a local parametrisation of M, then

$$\Psi : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow TM \subseteq M \times \mathbb{R}^m$$
$$(\bar{x}, \bar{v}) \longmapsto (\psi(\bar{x}), D_{\bar{x}} \psi(\bar{v}))$$

is a local parametrisation of the manifold TM.

- The above parametrisation has transition functions

$$\Psi_j^{-1} \circ \Psi_i : \mathcal{U} \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \qquad \mathcal{U} = \Psi_i^{-1}(\Psi_j(\mathbb{R}^n)) \subseteq \mathbb{R}^n$$
$$(\bar{x}, \bar{v}) \mapsto (\varphi_j^{-1} \circ \varphi_i(\bar{x}), \underbrace{D_{\bar{x}}(\psi_j^{-1} \circ \psi_i}_{\text{chain rule}}(\bar{v})))$$

that are <u>linear</u> isomorphisms in the 2^{nd} component $\mathbb{R}^n$
(but the lin. map depends on $\bar{x}$)

- <u>Key feature</u> $f \in C^\infty(M, N)$ induces a well-defined map

$$\begin{tikzcd} TM \arrow[r, two heads, "Tf"] \arrow[d, "p_{TM}"'] \arrow[dr, phantom, "\circlearrowleft"] & TN \arrow[d, "p_{TN}"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

which in the local parametrisations take the form

$$(\Psi^N)^{-1} \circ Tf \circ \Psi^M(\bar{x}, \bar{v}) = ((\psi^N)^{-1} \circ f \circ \psi^M(\bar{x}), D_{\bar{x}}((\psi^N)^{-1} \circ f \circ \psi^M)(\bar{v}))$$

Fibre Bundles

Let B and F be topological spaces / manifolds

Fibre bundles are particular "families" of spaces $\cong F$ (fibres) "parametrised" by the base B. More precisely, a fibre bundle consists of

- $p : E \underset{\text{surj.}}{\twoheadrightarrow} B$
- an open cover $\{U_i\}$ of B
- existence of isom. $\Phi_i : p^{-1}(U_i) \xrightarrow{\cong} U_i \times F$ so-called local trivialisations

subject to

$$\boxed{\begin{tikzcd} \Phi_i : p^{-1}(U_i) \arrow[r, "\cong"] \arrow[d, "p"'] & U_i \times F \arrow[dl, "\mathrm{proj}_{U_i}"] \\ U_i \arrow[ur, phantom, "\circlearrowright"] & \end{tikzcd}}$$

OBS: The data consists of $(\underset{\text{total space}}{E}, \underset{\text{base}}{B}, \underset{\text{Bundle proj.}}{p}, \underset{\text{fibre}}{F})$

Rmk
- The so-called fibre $p^{-1}(b) \subseteq E$ over $b \in B$ satisfies

  $p^{-1}(b) \cong F$ (All fibres are isomorphic), but:

  ⚠ This identification depends on the local trivialisation (not canonical!)

- If
$$\begin{tikzcd} E \arrow[r, phantom, "\cong"] \arrow[d, "p"'] & B \times F \arrow[dl, "\mathrm{proj}_B"] \\ B & \end{tikzcd}$$
then we say that (E, B, F, p) is (globally) trivial

- If we in addition fix the following data:
  - a subgroup $G < \mathrm{Diff}^\infty(F)$ called the structure group
  - a subset $H_b := \{ f : F \xrightarrow{\cong} p^{-1}(b) \}$ of diffeomorphisms on which G acts transitively by pre-composition: $\boxed{\{f\} \circ G = H_b}$ for any $f \in H_b$

and require that $\Phi_i^{-1} : U_i \times F \to p^{-1}(U_i)$ satisfy $\Phi_i^{-1}(b, \sqcup) \in H_b$ for all b

(in particular: $\Phi_j \circ \Phi_i^{-1}(b, y) = (b, g_b(y))$ with $g_b \in G < \mathrm{Diff}^\infty(F)$)

The data $(E, B, p, F, G, \{H_b\})$ defines a fibre bundle w. structure group G.

$$\begin{array}{ccccc} E & B & p & F & G \\ \end{array}$$

Ex $(TM, M, p_M, \mathbb{R}^n, Gl_n, \{H_b\})$ is a fibre bundle w. structure group $Gl_n(\mathbb{R})$

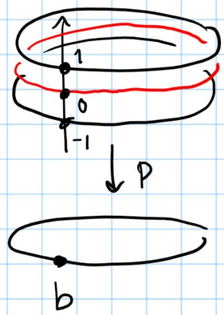
the linear identifications $\mathbb{R}^n \to p_{TM}^{-1}(b) \subseteq \mathbb{R}^m$

Ex
- $TS^1 \cong S^1 \times \mathbb{R}$ is globally trivial
- TS^2 is not globally trivial (as we will see later)

Ex $B = S^1$, $F = \mathbb{R}$

trivial bundle

$p^{-1}(b) = \mathbb{R}$

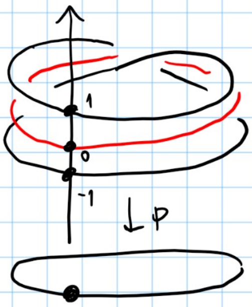


$E = B \times \mathbb{R} = S^1 \times \mathbb{R}$ (cylinder)

$B = S^1$

non-trivial bundle

(loc. trivial)



$E =$ Möbius band $\not\cong$ cylinder

$B = S^1$

Note the second $\mathbb{R}$-bundle can be given structure group
$$G = Gl_1(\mathbb{R}) < \mathrm{Diff}^\infty(\mathbb{R})$$
but not $Sl_1(\mathbb{R}) < Gl_1(\mathbb{R})$!

Morphisms of bundles with fibre F (and structure group G)

$$\begin{tikzcd} E_1 \arrow[r, "\Psi"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "\psi"'] & B_2 \end{tikzcd}$$

subject to $\boxed{\Psi \circ H_b^1 = H^2_{\psi(b)}}$ $\Rightarrow$ fibre-wise isomorphism

(where $H_x^i \subseteq \mathrm{Diff}^\infty(F, p_i^{-1}(x))$ is the subsets of preferred diffeomorphisms)

Or, formulation in local triv.: $\Phi^2 \circ \Psi \circ (\Phi^1)^{-1}(b, y) = (\psi(b), g_b(y))$

$$\boxed{g_b \in G < \mathrm{Aut}(F)}$$

We will develop tools for analysing bundles / isom.