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Principal bundles

A right G-action for a (topological/Lie) group G is a (cont./smooth) map
$$E \times G \to G \qquad (x\cdot g)\cdot h = x\cdot \underbrace{(g\cdot h)}_{\in G} \qquad (\Rightarrow x\cdot e = x)$$
$$(x, g) \mapsto x\cdot g \qquad \text{with } g, h \in G$$

$E/G := \{x\cdot G\}_{xG}$ = set of G-orbits $\qquad p: E \twoheadrightarrow E/G$, $x \mapsto x\cdot G$

If E has a right G-action which is free ($x\cdot g = x \Leftrightarrow g = e$) then the quotient projection $p: E \twoheadrightarrow E/G$ has fibres that all satisfy $p^{-1}(b) \cong G$.

Indeed: for any $x \in p^{-1}(b)$, the fibre over $b \in E/G$ is the orbit parametrised by
$$\varphi_{b,x}: G \xrightarrow{\cong} p^{-1}(b)$$
$$g \mapsto x\cdot g$$

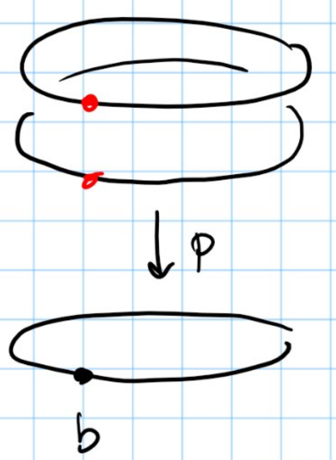
$\left(y = x\cdot g_y \Rightarrow x = y\, g_y^{-1}\right)$

Note $G \overset{\text{subgroup}}{\hookrightarrow} \mathrm{Diff}^\infty(G)$ by $g \mapsto (h \mapsto \ell_g(h) := g\cdot h)$ left multiplication $\qquad \varphi_{b,y}^{-1} \circ \varphi_{b,x}(g) = \ell_{g_y^{-1}}$

Def A Principal G-bundle is a space E equipped with a right G-action for which $(E, B = E/G, p, F = G, G, \{\mathcal{H}_b = \{\varphi_{b,x}\}\})$ is a fibre bundle.

Ex $B = S^1$, $F = \mathbb{Z}_2$ $\qquad p^{-1}(b) \cong \mathbb{Z}_2$

trivial bundle



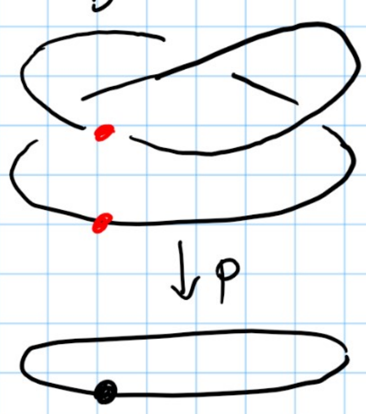
$E = S^1 \sqcup S^1 \cong S^1 \times \mathbb{Z}_2$

$\downarrow p$

$B = S^1$

non-trivial bundle

(loc. trivial)



$E \cong \partial(\text{Möbius band}) = S^1 \ncong S^1 \times \mathbb{Z}_2$

$\downarrow p$

$B = S^1$

Lie groups

To denote smooth principal bundles we need the concept of smooth groups i.e. Lie groups

Def A smooth manifold G is a Lie group if there exists smooth maps $m : G \times G \to G$, $(g_1, g_2) \mapsto g_1 \cdot g_2$, $\mathrm{inv} : G \to G$, $g \mapsto g^{-1}$ and an element $e \in G$

for which:
- $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$ (associativity)
- $e \cdot g = g = g \cdot e$ (neutral element)
- $\mathrm{inv}(g) \cdot g = g \cdot \mathrm{inv}(g) = e$ (invertibility)

} usual group axioms

Let $\mathrm{Diff}^\infty(M)$ be the group of diffeomorphisms of a mfd M, i.e. φ & $\varphi^{-1} \in C^\infty(M, M)$

Exercise 10 Write $l_g(x) = g \cdot x$, $r_g(x) = x \cdot g$. Show that

$\mathrm{inv}, l_g, r_g \in \mathrm{Diff}^\infty(G)$, and that moreover

$G \to \mathrm{Diff}^\infty(G)$, $g \mapsto l_g$ and $G^{op} \to \mathrm{Diff}^\infty(G)$, $g \mapsto r_g$ are inclusions of groups.

Ex
- Countable groups are 0-dim Lie groups when endowed with the discrete topology (locally homeomorphic to $\mathbb{R}^0$)
- $Gl_n(\mathbb{R})$, $Gl_n(\mathbb{C})$ (open subsets of $\mathbb{R}^{n^2}$, $\mathbb{C}^{n^2}$)
  n^2-dim, $4n^2$-dim
- $Sl_n = \{A \mid \det A = 1\} < Gl_n$ closed subgroups (det is continuous)
  $\dim Sl_n(\mathbb{R}) = n^2 - 1$, $\dim Sl_n(\mathbb{C}) = 4n^2 - 2$
- $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ compact Lie groups (hence closed in Gl_n)
  dim: $\frac{n(n-1)}{2}$, $\frac{n(n-1)}{2}$, n^2, n^2

Thm A subgroup $H < G$ of a Lie group that is closed is itself a Lie group (smooth mfd, ...) and the inclusion is smooth; i.e. H is a Lie subgroup.

<u>Equivalent</u> formulation of the structure of a principal G-bundle $p: E \to B$

- $\exists$ local trivialisations $\Phi_i : p^{-1}(U) \to U \times G$ w. transition functions

$$\boxed{\Phi_j \circ \Phi_i^{-1}(u, g) = (u, \underbrace{g_u \cdot g}_{\ell_{g_u}(g)})} \qquad g_u, g \in G \qquad u \in U$$

- Right G-action on E is given by

$$\boxed{\Phi_i(x) = (u, h) \iff \Phi_i(x \cdot g) = (u, h \cdot g)}$$

We typically construct principal bundles via <u>homogeneous spaces</u>.

M smooth manifold, G Lie group

$M \times G \to M$ smooth G-action from the <u>right</u>
$(m, g) \mapsto m \cdot g$

$M/G = \{m \cdot G\}_{m \in M}$ cosets endowed with quotient topology $M \twoheadrightarrow M/G$

⚠ M/G is not necessarily a manifold

- For any $m \in M$

$G_m := \{g \mid m \cdot g = m\} < G$ <u>stabiliser</u> subgroup $m \in M$, which is closed

$\Delta \subseteq M \times M$ closed $\downarrow$ closed

[Thm] $\Rightarrow$ G_m Lie subgroup

For $m' = m \cdot h$: $G_{m'} = h^{-1} \cdot G_m \cdot h$ a conjugate subgroup ($\Rightarrow G_m \cong G_{m'}$)
$h \in G$

- G acts <u>transitively</u> on M if $m \cdot G = M$ for some (and hence all) $m \in M$

i.e. $M/G = pt$. Such M is called a <u>G-homogeneous space</u>

Technical Rmk [Sard's thm] implies that, for a homogeneous space

$\varphi : G \twoheadrightarrow M$, $g \mapsto m\cdot g$

- $\dim M \leq \dim G$ (since φ surj.)
- φ is submersive at all points (since it is submersive at some point)

It follows that p_m is <u>open</u> (i.e a quotient map)

In fact, next we will see that this is a <u>smooth principal G_m-bundle</u>.

Thm 2.4 from [Sharpe; Differential Geometry: Cartan's Generalisation of Klein's Erlangen Program] then gives

[Thm] $\Rightarrow$ H Lie subgroup

Thm 1.) For a <u>closed</u> subgroup $H < G$ of a Lie group, there exists a <u>unique</u> smooth structure on the quotient space $G/H = \{H\cdot g\}_{g\in G}$ for which

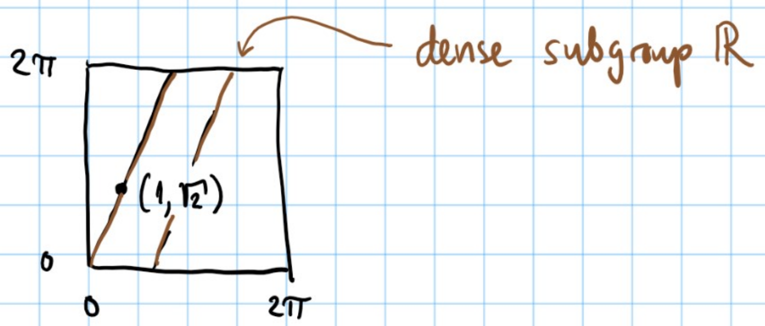
- $q_m : G \to G/H$ (canonical proj.) is smooth
- $G/H \times G \to G/H$, $(H\cdot h, g) \mapsto H\cdot(g\cdot h)$ makes G/H into a G-homogeneous space $(G_{[e]} = H < G)$

2.) If the manifold M is a homogeneous G-space then, for any $m\in M$, $p_m : G \twoheadrightarrow M$, $g \mapsto m\cdot g$ is a smooth principal G_m-bundle, and there is an isomorphism

$$\begin{array}{ccc} G & \xrightarrow{id} & G \\ p_m \downarrow & & \downarrow q_m \\ M & \xrightarrow[\cong_{C^\infty}]{} & G/G_m = \{g\cdot G_m\} \\ m & \longmapsto & [e] \end{array}$$

of principal G_m-bundles.

Rmk $H<G$ being <u>closed</u> is crucial, there are (non-closed) subgroups $\mathbb{R} < S^1\times S^1$ whose quotient is not a manifold.



There are plenty of important examples $G \twoheadrightarrow G/H$ where $H \overset{\text{closed}}{<} G$

Ex 1.) The sphere S^n is a homogeneous space

$SO(n+1)$ acts transitively on $S^n \subseteq \mathbb{R}^{n+1}$

The stabilizer of $(0,..,0,1) \in S^n$ is $SO(n+1)_{(0,..,0,1)} = SO(n) < SO(n+1)$

$$\rightsquigarrow \quad \underset{n(n-1)/2}{SO(n)} \hookrightarrow \underset{(n+1)\cdot n/2}{SO(n+1)} \twoheadrightarrow \underset{n}{S^n} \qquad SO(n)\text{-principal bundle}$$
dim

$$O(n) \hookrightarrow O(n+1) \twoheadrightarrow S^n \qquad O(n)\text{-principal bundle}$$

2.) The projective plane $\mathbb{R}P^n$ is a homogeneous space

$O(n) < O(n+1)$ closed $\Rightarrow$ $(\pm 1)\cdot O(n) < O(n+1)$ also closed

$\boxed{\mathbb{R}P^n := O(n+1)/(\pm 1)\, O(n)}$ smooth n-dim mfd. by above thm

$$\underset{\frac{n\cdot(n-1)}{2}}{(\pm 1) O(n)} \hookrightarrow \underset{\frac{(n+1)\cdot n}{2}}{O(n+1)} \twoheadrightarrow \underset{n}{\mathbb{R}P^n} \qquad (\pm 1) O(n)\text{- principal bundle}$$
dim

$\big((\pm 1)\cdot id < O(n)$ central subgroup$\big)$

3.) S^{2n+1} is also a $U(n+1)$-homogeneous space

$U(n+1) \circlearrowright S^{2n+1} \subseteq \mathbb{C}^{n+1}$ trivializes action with stabiliser $U(n+1)_{(0,0,..,0,1)} = U(n) < U(n+1)$

$$\underset{n^2}{U(n)} \hookrightarrow \underset{(n+1)^2}{U(n+1)} \twoheadrightarrow U(n+1)/U(n) = S^{2n+1}$$
dim

4.) The complex projective plane is a smooth manifold constructed analogously

$S^1 \cdot U(n) < U(n+1)$ closed subgroup $\big(S^1 \cdot id < U(n+1)$ central$\big)$

$\boxed{\mathbb{C}P^n := U(n+1)/S^1 \cdot U(n)}$ smooth $2n$-dimensional manifold

$$\underset{n^2+1}{S^1 \cdot U(n)} \hookrightarrow \underset{(n+1)^2}{U(n+1)} \twoheadrightarrow \underset{2n}{\mathbb{C}P^n}$$
dim