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Exercise 11 Show that, if $K < H < G$ are closed subgroups of a Lie group G, and $K < H$ moreover is <u>normal</u> ($\Leftrightarrow H/K$ is a Lie group), then there exists a smooth right H/K-action on G/K that makes
$G/K \twoheadrightarrow G/H$ a smooth H/K-principal bundle with G/H a manifold.

<u>Hint:</u> argue in local trivialisations

Exercise 12 Show that the following quotients are principal bundles for the natural actions

- $S^n \twoheadrightarrow S^n/\mathbb{Z}_2 = \mathbb{R}P^n \qquad \mathbb{Z}_2 = \{\pm \mathrm{Id}\}$
- $\mathbb{R}^{n+1} \setminus \{0\} \twoheadrightarrow (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^* = \mathbb{R}P^n$

  (Note: $\mathbb{R}^{n+1} \setminus \{0\} = Gl_{n+1}(\mathbb{R}) / \{A \in Gl_{n+1}(\mathbb{R});\ Ae_1 = e_1\}$
- $S^{2n+1} \twoheadrightarrow S^{2n+1}/S^1 = \mathbb{C}P^n$
- $\mathbb{C}^{n+1} \setminus \{0\} \twoheadrightarrow (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^* = \mathbb{C}P^n$

  (Note: $\mathbb{C}^{n+1} \setminus \{0\} = Gl_{n+1}(\mathbb{C}) / \{A \in Gl_{n+1}(\mathbb{C});\ Ae_1 = e_1\}$

<u>Hint</u>: you are free to use the previous exercise

<u>General question</u>: are these principal bundles trivial products Base × Fibre or "twisted products"?

Next: we will find tools to compute homotopy groups of these spaces and at least get partial insight.

Some preliminary results

Prop 1.) $\pi_0(G,e)$ is a group which can be identified with $G/_{[e]}$ where $[e] \triangleleft G$ is the normal subgroup given by the path-component of the unit $e \in G$.

($\pi_0(M,*)$ is typically just a set)

2.) $\pi_1(G,e)$ is abelian

3.) $\pi_k(X \times Y, *_X \times *_Y) \cong \pi_k(X,*_X) \times \pi_k(Y,*_Y)$

Exercise 13 Prove the Proposition.

If the fiber bundle $E \twoheadrightarrow B$ is trivial, i.e $E \cong B \times F$, then

$$\pi_k(E, *_B \times *_F) \cong \pi_k(B,*_B) \times \pi_k(F,*_F)$$

In particular, the inclusion $\iota : (F,*_F) \hookrightarrow (B\times F, *_B \times *_F)$, $y \longmapsto (*_B, y)$

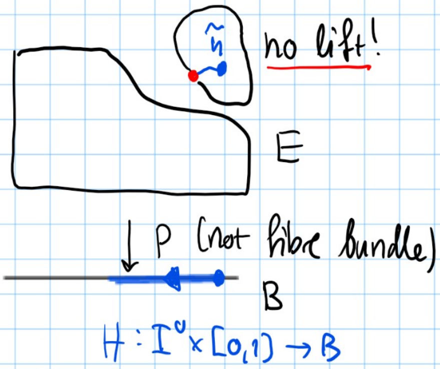
and projection $p : (B\times F, *_B \times *_F) \to (F, *_F)$

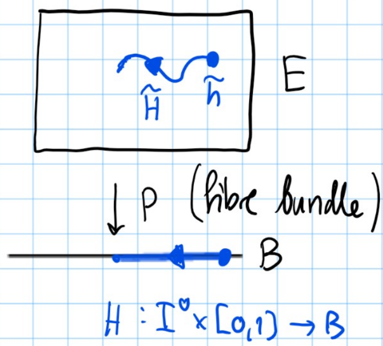
fit into a short exact sequence of groups

$$\boxed{0 \to \pi_k(F,*_F) \overset{\iota_*}{\hookrightarrow} \pi_k(B,*_B) \times \pi_k(F,*_F) \overset{p_*}{\twoheadrightarrow} \pi_k(B) \to 0}$$

Failure of injectivity of ι_* / surjectivity of p_* is an obstruction to triviality. A crucial tool for studying this is the homotopy lifting property for maps into fibre bundles.

Pictorially:





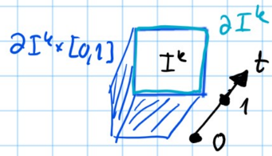
$I = [0,1]$, $\partial I^k = \bigcup_{i=1}^{k} [0,1]\times\dots\times\{0,1\}\times\dots\times[0,1]$ ← union of $2k$ cubes of dim $k-1$

Thm (The homotopy lifting property) If $\tilde h : I^k \to E$ is a continuous map from the k-cube into a fibre bundle (the "initial condition"), and $H: I^k \times [0,1]_t \to B$ is a homotopy from

$$H_0 = h := p\circ \tilde h : I^k \to B \qquad \text{to} \qquad H_1 : I^k \to B$$

then there exists a cont. lift for which $\tilde H_0 = \tilde h$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ I^k\times[0,1] \arrow[r, "H"'] \arrow[ur, dashed, "\tilde H"] & B \end{tikzcd}$$

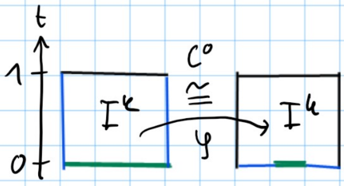


In addition

- if we already are given a partial lift $\tilde K : \partial I^k \times [0,1] \to E$, $\tilde K_0 = \tilde h|_{\partial I^k}$, then we can require that $\tilde H = \tilde K|_{\partial I^k\times[0,1]}$
- Two different lifts of the same H with the same initial condition $\tilde h$ are homotopic through lifts of the same type.

Proof Assume that we have proven the statement given a lift $\tilde K : \partial I^k \times [0,1] \to E$ over the boundary of the cube. Since there is a homeomorphism

$$\varphi : I^k \times [0,1] \xrightarrow{\cong} I^k \times [0,1]$$



which restricts to a homeomorphism

$$\partial I^k \times [0,1] \cup I^k \times \{0\} \xrightarrow{\cong} I^k \times \{0\}$$

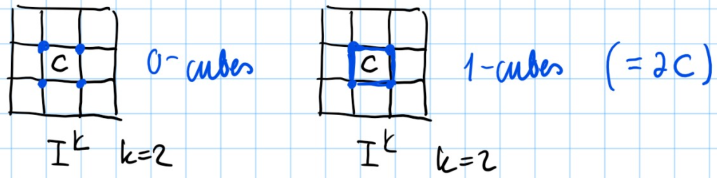
we can apply the result to the homotopy $H_t \circ \varphi$ and initial condition $\tilde h \circ \varphi$. Call the produced lift $\tilde{\tilde H}$, the sought lift is then $\tilde H := \tilde{\tilde H} \circ \varphi^{-1}$.

Assume by induction that the full statement has been shown for all l-cubes with $l < k$. We show that the relative statement holds for k-cubes.

Let $\{U_j\}$ be an open cover of B s.t. $\exists$ local trivialisations $\Phi_j : p^{-1}(U_j) \to U_j \times F$

1.) Subdivide the k-cube into smaller cubes $C_{M,\underline{i}} := [i_1, i_1 + 1/M]\times\dots\times[i_k, i_k + 1/M]$ for $M \gg 0$ s.t. $H|_{C_{M,\underline{i}}\times[i_0,\, i_0 + 1/M]}$ take values inside some $U_j \subseteq B$

2.) By induction we may construct the lift over all $\partial C_{M,\underline{i}} \times [0,1]_t$. Divide $\partial C_{M,\underline{i}}$ into a union of cubes of dim $0,1,2,\dots,k-1$ and lift over all subcube $\times [0,1]_t$



3.) We proceed to extend the partial lift constructed above to lifts of $H|_{C_{M,\underline{i}} \times [i_0, i_0+1/M]}$ over the interior $C_{M,\underline{i}} \times [i_0, i_0+1/M]$.

Start at the "bottom" of $I^k \times [0,1]$ $(i_0 = 0)$ and work upwards.

We have reduced the problem to constructing a lift $\tilde{H}$

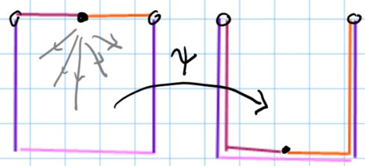
$$\begin{tikzcd} & U \times F \arrow[d, "proj_U"] \\ I^k \times [0,1] \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & U \end{tikzcd} \qquad \text{subject to } \tilde{H}|_0 = \tilde{h}$$

that extends a partial lift $\tilde{K} : \partial I^k \times [0,1] \longrightarrow U \times F$, $\tilde{K}_0 = \tilde{h}|_{\partial I^k \times \{0\}}$

We can take $\tilde{H} = (H, \tilde{K} \circ \psi)$ where $\psi : I^k \times [0,1] \xrightarrow{C^0} \underline{\partial I^k \times [0,1]} \cup \underline{I^k \times \{0\}}$

s.t. $\psi|_{\partial I^k \times [0,1]} = id$

(not a homeomorphism!)



The <u>base case</u> $k=0$ is easy: Any path in B can be be lifted to E (relative a choice of starting point for the lift) Again: use local coordinates.

Finally: Uniqueness of lift up to homotopy follows from lifting the trivial homotopy between H & H with a suitable initial condition.

$\square$

Long exact sequence of homotopy groups

Recall that $\to G_m \xrightarrow{f_m} G_{m-1} \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_2} G_1 \xrightarrow{f_1} G_0$ is said to be

exact if f_i are group homomorphisms that satisfy $\boxed{\ker f_i = \operatorname{im} f_{i+1}}$

Ex 1.) $0 \to G_i \xrightarrow{f_i} G_{i-1} \to \cdots$ exact $\Rightarrow$ f_i injective

2.) $\cdots \to G_i \xrightarrow{f_i} G_{i-1} \to 0$ exact $\Rightarrow$ f_i surjective

3.) $0 \to G_3 \hookrightarrow G_2 \twoheadrightarrow G_1 \to 0$ exact $\Leftrightarrow$ $G_1 = G_2 / G_3$

The latter is called a short exact sequence

E.g. homogeneous G-spaces with normal stabiliser subgroup $G_x \triangleleft G$ (the space is itself a group, not merely a quotient)

$$0 \to G_x = H \hookrightarrow G \twoheadrightarrow G/H \to 0$$

Thm For any fibre bundle $p: E \twoheadrightarrow B$ with a choice of basepoints $*_B \in B$ and $* \in F := p^{-1}(*_B) \subseteq E$ there is a long exact sequence of homotopy groups:

$$\boxed{\cdots \to \pi_{k+1}(B, *_B) \xrightarrow{\delta_{k+1}} \pi_k(F, *) \xrightarrow{\iota_*} \pi_k(E, *) \xrightarrow{p_*} \pi_k(B, *_B) \xrightarrow{\delta_k} \pi_{k-1}(F, *) \to \cdots}$$

where $\iota : (F, *) \hookrightarrow (E, *)$ & $p : (E, *) \twoheadrightarrow (B, *_B)$ are the canonical inclusion and projection, while δ_{k+1} is constructed as follows:

i) Represent any $[\gamma] \in \pi_{k+1}(B, *_B)$ by $\gamma \in C(I^{k+1}, B)$, $\gamma|_{\partial I^{k+1}} \equiv cst_{*_B}$

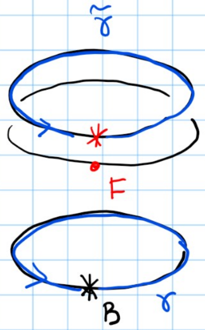
ii) Lift γ partially to $\tilde{\gamma} : \partial I^k \times [0,1] \cup I^k \times \{0\} \to \{*\} \subseteq E$

iii) Homotopy lifting theorem $\Rightarrow$ lift extends to $\tilde{\gamma} : I^k \times [0,1] \to B$

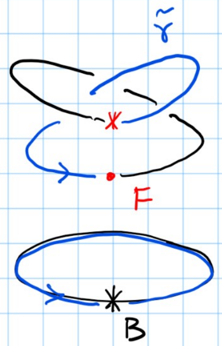
$$\boxed{\delta_{k+1}[\gamma] = \left[\tilde{\gamma}|_{I^k \times \{1\}}\right]}$$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ \underbrace{I^k \times [0,1]}_{I^{k+1}} \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & B \end{tikzcd}$$

Illustration of $\delta : \pi_1(B, *_B) \to \pi_0(F, *)$



$\delta[\gamma] = [*] \in \pi_0(F)$



$\delta[\gamma] \neq [*] \in \pi_0(F)$

$\Rightarrow$ non-trivial bundle!

Rmk The three right-most terms

$$\to \pi_1(B, *_B) \xrightarrow{\delta_1} \pi_0(F, *) \xrightarrow{\iota_*} \pi_0(E, *) \xrightarrow{p_*} \pi_0(B, *_B) \to 0$$

need <u>not</u> be groups. However, they are sets with a <u>preferred element</u> (the path compnent of the basepoint).
Exactness works analogously: the kernel in the preimage of the preferred element.