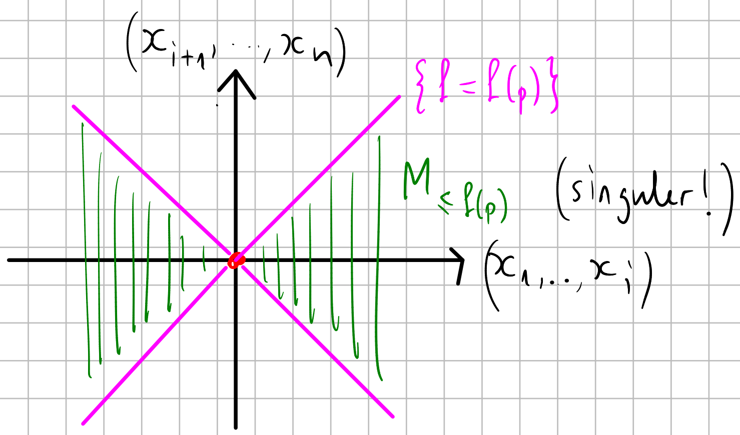
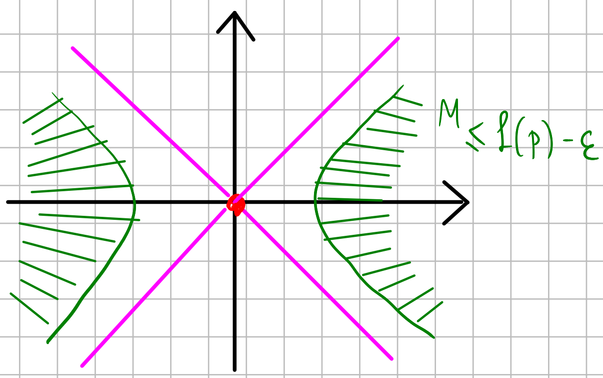
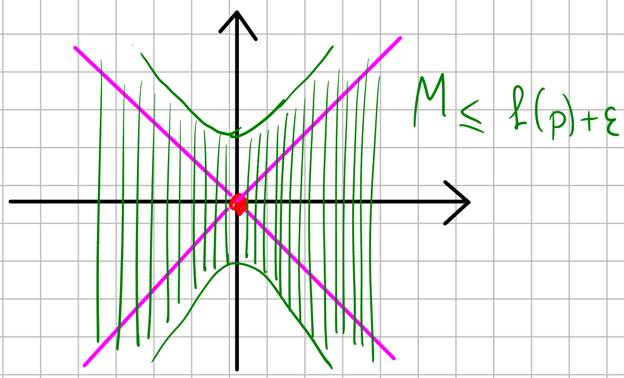
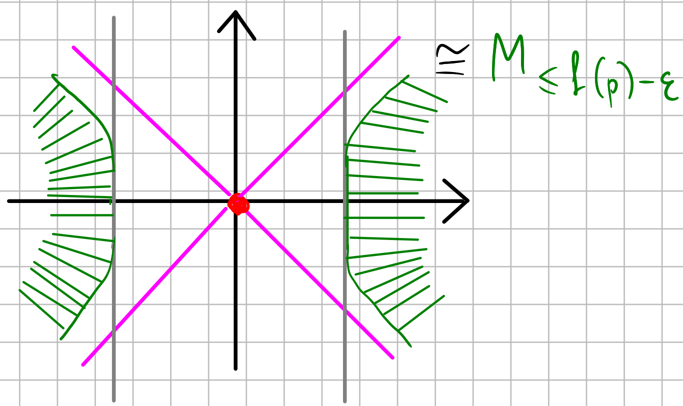
The change of topology at a critical value

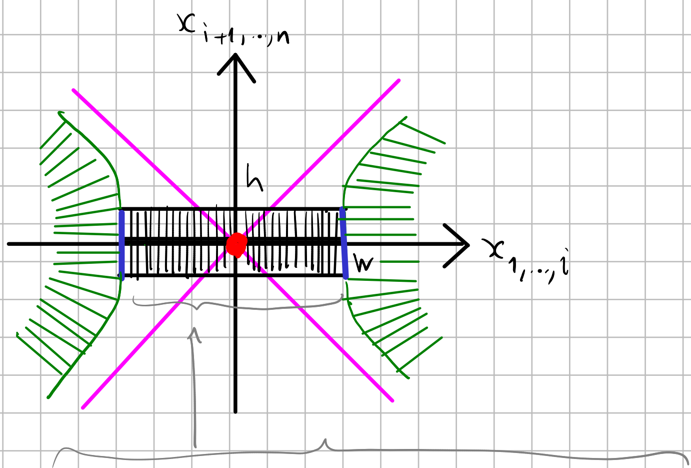


$$f(\bar{x}) = f(p) - (x_1^2 + \dots + x_i^2) + (x_{i+1}^2 + \dots + x_n^2) \quad \text{(Morse lemma)}$$









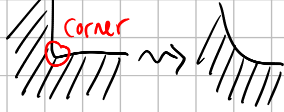
attachment of <u>i-dimensional handle</u> $= \{x_1^2 + \dots + x_i^2 \leq w^2,\ x_{i+1}^2 + \dots + x_n^2 \leq h^2\}$

$= D^i \times D^{n-i}$

$$\text{i-handle} \cap \partial(M_{\leq f(p)-\varepsilon}) = \boxed{\partial D^i \times D^{n-i}} = S^{i-1} \times D^{n-i}$$

fattened / framed $(i-1)$-dim sphere

$M_{\leq f(p)-\varepsilon} \cup$ i-handle has corners, but can be smoothed to yield $M_{f(p)+\varepsilon}$



So: The manifold $M_{\leq f(p)+\varepsilon}$ is determined by the embedding $S^{i-1} \times D^{n-i} \hookrightarrow \partial(M_{\leq f(p)-\varepsilon})$ of a "framed $(i-1)$-sphere" along which the handle is glued.

Classification of surfaces (dim $M=2$)

After cancelling max^s / min^s ($i=2$ / $i=0$) and saddles ($i=1$): We can assume that max and min are unique.

Assume that $f: \Sigma \to \mathbb{R}$ is a Morse function on a surface with a unique max & min.

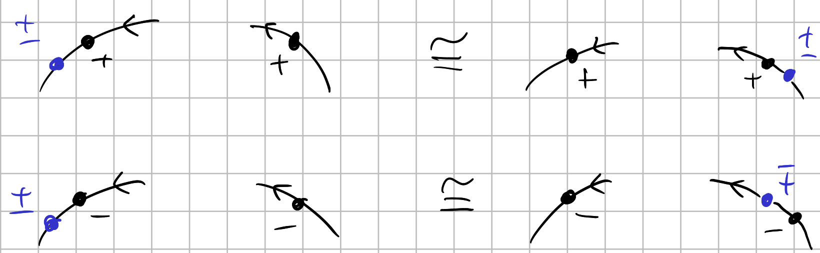
$\Sigma_{\leq \max - \varepsilon}$ is determined by the following data:

A (possibly zero) number of embeddings of 0-spheres in ∂D^2, where each sphere is marked either as $+$ or $-$.

1-handle attachments determined by $S^0 \hookrightarrow \partial D^2$

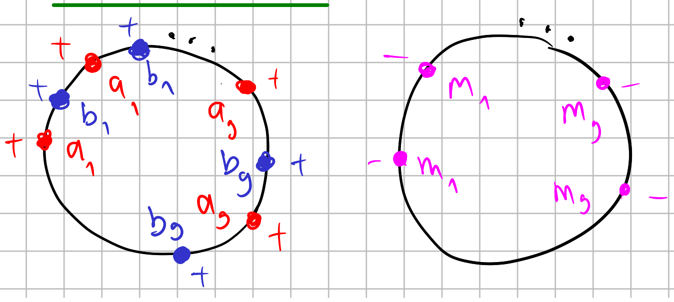
Exercise 23

Show with pictures that



Exercise 24

Use the above to find a normal form for $\Sigma_{\leq \max - \varepsilon}$



(Hint: the boundary must be connected after the 1-handle attachments)

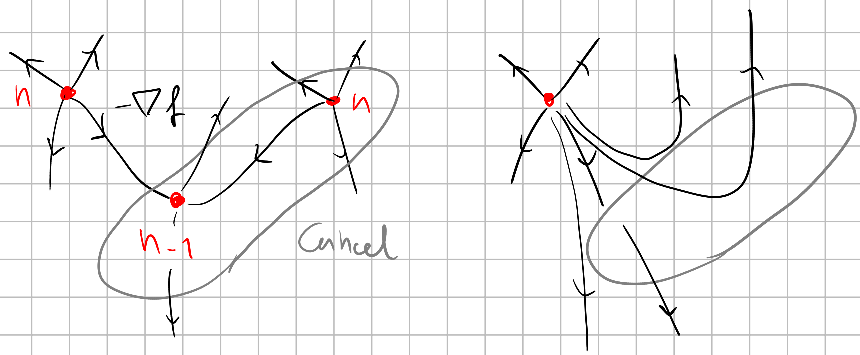
Morse theory provides descriptions of manifolds as a <u>sequence of handle attachments</u>.

Lem Any Morse function $f: M \to \mathbb{R}$ on a connected compact manifold can be deformed so that:

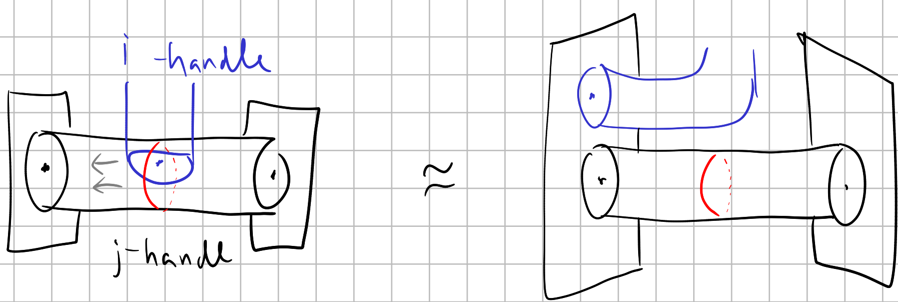
1.) max & min are unique

2.) the critical points have unique critical values and the index of the critical points are non-decreasing with respect to the order induced by f.

<u>Idea of proof</u> 1.)



2.) an i-handle can be assumed to be attached to the complement of all j-handles w. $j \geq i$. (Attached along a $(i-1)$-dim sphere



Observe: j-handle $\cong D^j \times D^{n-j}$, it suffices to disjoin the attaching region for the new handle from $\{0\} \times S^{n-j-1} \subseteq \partial(D^j \times D^{n-j})$

Since $i-1+n-j-1 < n$ when $j \geq i$, this is generically the case

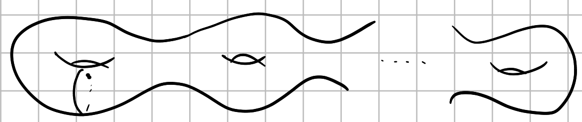
Exercise 25 Realize Σ_g as the level set $\{f=c\}$ of a Morse function $f: M_{\leq c} \to \mathbb{R}$ with precisely $1+g$ critical points in $M_{\leq c}$, $\partial(M_{\leq c}) = \Sigma_g$.

In fact, any orientable $M_{\leq c}$ where $f: M_{\leq c} \to \mathbb{R}$ has critical points consisting of a unique minimum & g nr of 1-handles are determined up to diffeomorphism by $g \geq 0$; they are called genus g handlebodies.

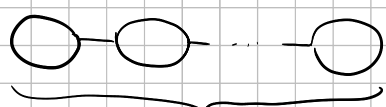
Exercise 26 Compute $\pi_1(M_{\leq c})$ for a handle body of genus g.

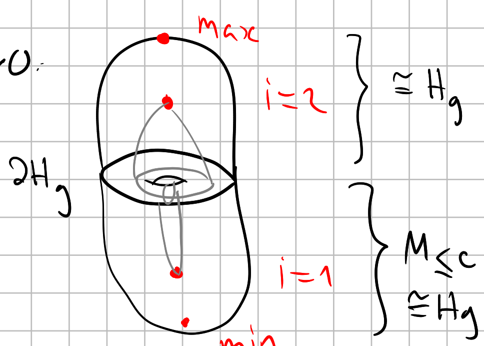
The above lemma implies that any closed orientable 3-dim manifold M^3 can be obtained as a union $M = M_{\leq c} \cup (M \setminus M_{\leq c})$ of two "handle bodies" H_g of genus $g \geq 0$.

H_g



$\partial H_g = \Sigma_g$ = nbhd in $\mathbb{R}^3$ of



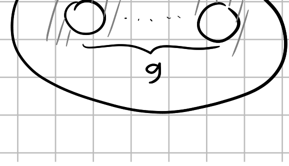


"Heegaard splitting"

(Replace f by $-f$ to see that $M \setminus M_{\leq c}$ also is a handle body of the same genus)

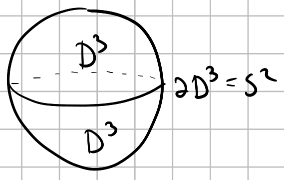
Heegaard Diagrams

To construct M from $M_{\leq c} \cong H_g$ we just need to attach the 2-handles; they are determined by an embedding of #g disjoint simple closed curves in $\partial H_g \cong \Sigma_g$ with the property that $\Sigma_g \setminus$ curves $\cong$

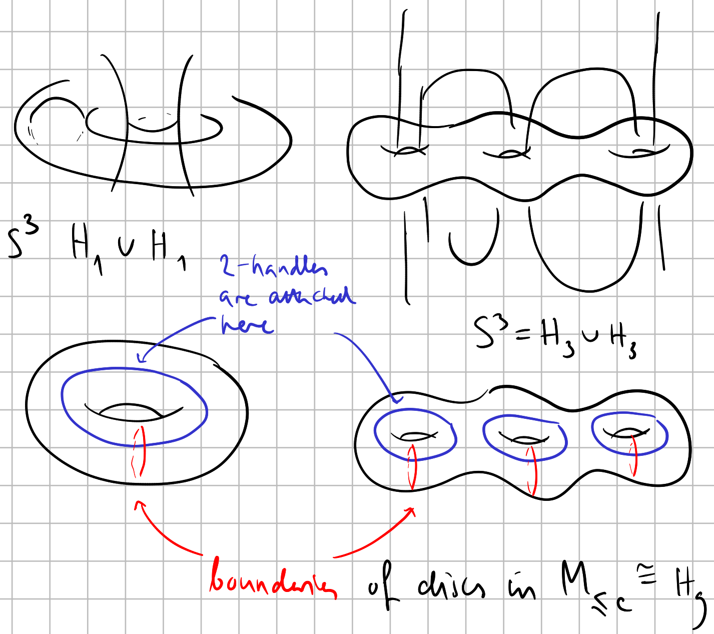


This is the Heegaard diagram (when $g>0$)

Ex 3-sphere



$H_0 = D^3$



[All orientation-preserving automorphisms are isotopic to id_{S^2} by [Smale], we do not need any diagrams here]

Exercise 27 Show that any Heegaard diagram on $\mathbb{T}^2$ gives rise to the total space of an S^1-bundle over S^2.

Hint: $H_1 \cong D^2_{(r,\varphi)} \times S^1_\theta$ is a trivial S^1-bundle with fibres given by $\{\varphi = k\theta\}$, $k \in \mathbb{Z}$,

Two complicated phenomena:
- topology of 3-dim manifolds
- diffeomorphism groups of surface of genus $g \geq 2$.

Next we will connect this to a third phenomenon:

- Embeddings of 1-spheres in S^3 (knot theory)