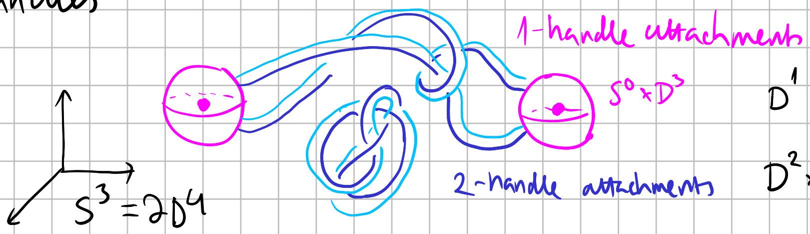
4-dim manifolds

Can be built by attaching handles of index $\underbrace{0}_{D^4}$ $\overbrace{\ 1}^{\text{handlebody}}$ 2 $\overbrace{3\ 4}^{\text{handlebody}}$

Laudenbach-Poenaru '70 Any closed M^4 is determined by its 1 & 2-handles

Kirby diagrams encode the framed attaching spheres in $\partial D^4 = S^3$ for the 1 & 2-handles



$D^1 \times D^3 \supseteq S^0 \times \{0\}$

$D^2 \times D^2 \supseteq S^1 \times \{0\}, S^1 \times \{1\}$

Recall • 1-handles are determined by embeddings of $S^0 \times D^3 \subseteq D^1 \times D^3$

• 2-handles are determined by embeddings of $S^1 \times D^2 \subseteq D^2 \times D^2$

Up to relevant identifications, these embeddings are determined by the core and a push-off along the framing, i.e. $S^1 \times \{0\}, S^1 \times \{1\} \subseteq S^1 \times D^2$

Thm (Lickorish-Wallace '60s) Any closed conn. orientable 3-dimensional manifold N^3 can be realised as the boundary $N = \partial M^4$ of a 4-dimensional manifold, where M admits a Morse function which
- is constant along N
- has a unique minimum in $M \setminus \partial M$
- remaining crit points are all of index $= 2$

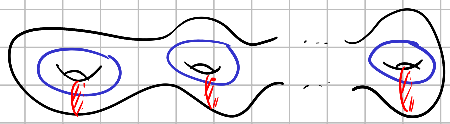
} M is obtained by 2-handle attachments on D^4

This gives a link between: topology of 3-dim manifolds and (surgery on) framed knots / links in S^3

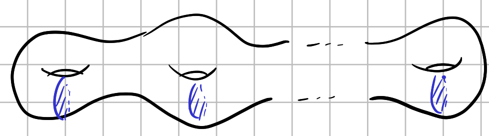
see below

We will study knots further in part §III

Idea of proof Start with a genus-g Heegaard Splitting for $S^3 = \partial D^4$.



glue boundaries of the handle-bodies



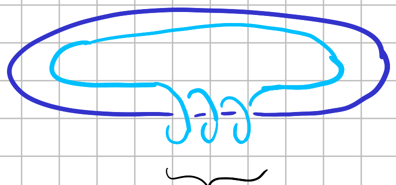
Attach 1-handles on B^4.

Same surface, but different Heegaard Diagram.

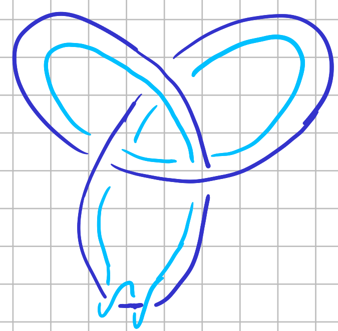
3-mfd obtained by removing nbhd $S^1 \times D^2$ of — and gluing back in $D^2 \times S^1$

$S^1 \times \{pt\}$ $\square$

Ex



#k twists



$k=0$: $S^1 \times S^2$

$k=1$: S^3

$k=2$: $\mathbb{R}P^3$

Poincaré homology sphere

Total space of S^1-bundle over S^2 with clutching function $\varphi_k : \partial D^2 \xrightarrow[k:1]{} S^1$ given by $y \mapsto k \cdot y$.

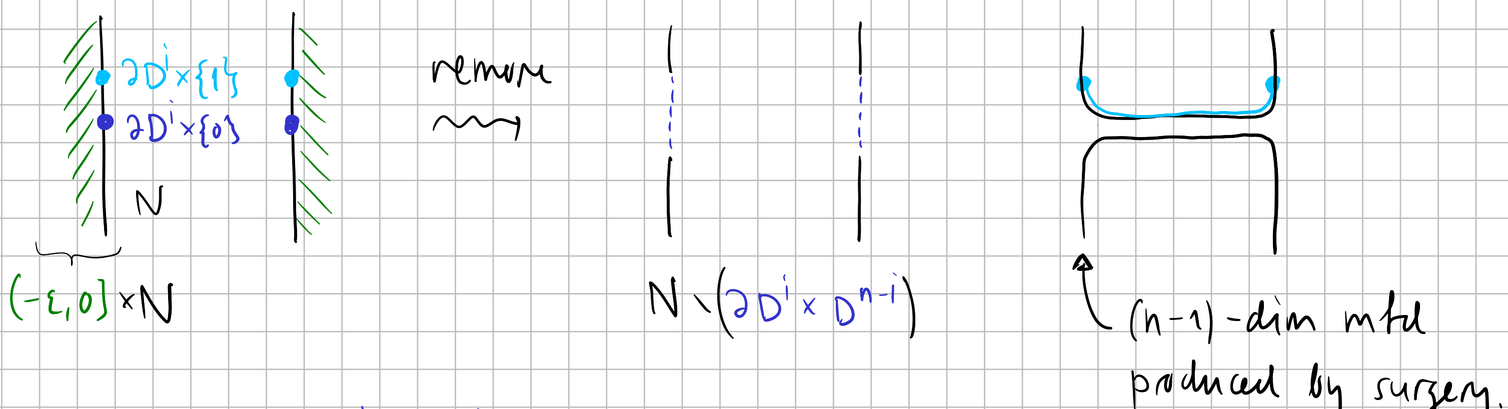
Obs: The push-off of the core along the framing bounds a disc inside the 3-manifold obtained.

Surgery: The process of deforming the boundary of D^4 by the partial boundaries of handle-attachments, i.e. the deformation

$$\partial(M_{\leq f(p)-\varepsilon}) \rightsquigarrow \partial(M_{\leq f(p)+\varepsilon}) \quad \text{where } p \text{ is a critical point}$$

of index $= i$ (corr. to an i-handle) is called surgery on a framed $(i-1)$-sphere, or simply $(i-1)$-surgery.

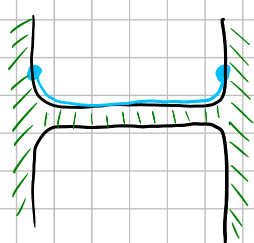
Any $(n-1)$-dim. manifold N^{n-1} can be realised as the boundary of a n-dim manifold $(-\varepsilon, 0] \times N$



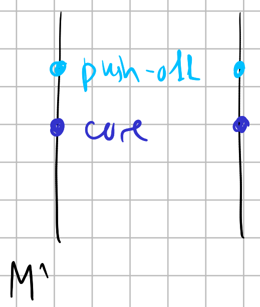
① Remove $\partial D^i \times D^{n-i} \subseteq N$ (solid torus if $i=2, n=4$)

② Glue back $D^i \times \partial(D^{n-i})$ s.t. $D^i \times pt$ bounds the sphere $\partial D^i \times \{1\} \subseteq N$ obtained by pushing $\partial D^i \times \{0\}$ along the framing. (again a solid torus if $i=2, n=4$)

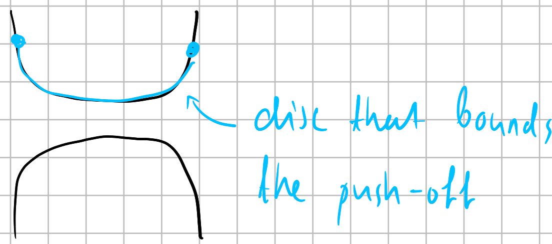
The result is the same as the new boundary after the corresponding i-handle attachment on $(-\varepsilon, 0] \times N$.

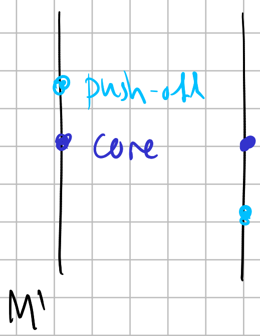


Ex Some surgeries that are possible to draw.

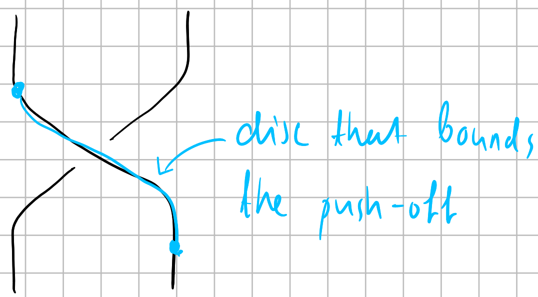


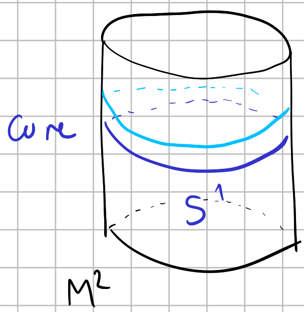
(0-) surgery on $S^0 \subseteq M^1$ $\rightsquigarrow$



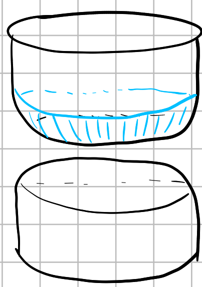


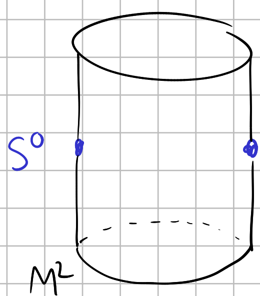
(0-) surgery on $S^0 \subseteq M^1$ $\rightsquigarrow$



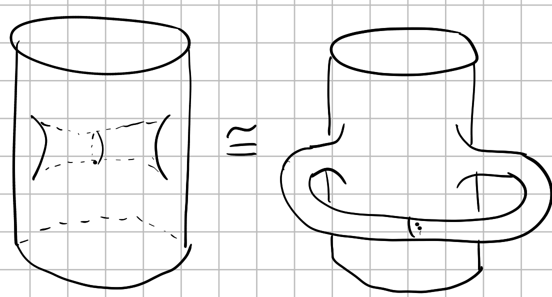


(1-) surgery on $S^1 \subseteq M^2$ $\rightsquigarrow$





(0-) surgery on $S^0 \subseteq M^2$ $\rightsquigarrow$



Invariants from Morse functions

A chain complex:

$$\dots \to C_i \xrightarrow{\partial} C_{i-1} \xrightarrow{\partial} C_{i-2} \to \dots \quad \partial^2 = 0, \qquad C_i: \ \mathbb{k}\text{-vector spaces } \underline{\text{or}} \text{ abelian groups}$$

$$H_i(C_\bullet) = \underbrace{\ker(\partial|_{C_i})}_{\text{cycles}} \Big/ \underbrace{\partial(C_{i+1})}_{\text{exact cycles, or "boundaries"}}$$ <u>homology</u> of the chain complex

$$C_i^* := \mathrm{Hom}_{\mathbb{k}}(C_i, \mathbb{k}) \qquad \big(\mathrm{Hom}_{\mathbb{Z}}(C_i, \mathbb{Z}) \text{ in the general case}\big)$$

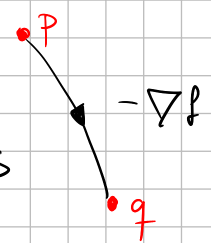
$$d := \partial^* : C_i^* \to C_{i+1}^*, \qquad d^2 = 0 \qquad \text{"co-complex"}$$

The Morse homology complex

$f: M \to \mathbb{R}$ a Morse function which is <u>proper</u> & <u>bounded from below</u> (automatic when M is compact)

$$C_i^{\mathrm{Morse}}(f) := \bigoplus_{\substack{p \in \mathrm{Crit}(f) \\ \text{index } p = i}} \mathbb{Z} \cdot p \qquad \Big(\text{or } \bigoplus \mathbb{k} \cdot p; \text{ here we take } \mathbb{k} = \mathbb{Z}_2\Big)$$

coefficient in front of "q" in expression "∂p"

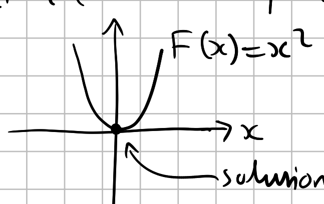


$\langle \partial p, q \rangle$ = (signed) count of negative gradient flow-lines from p to q for which $\mathrm{index}(p) = \mathrm{index}(q) + 1$, for a metric g on M for which these flow-lines are <u>regular</u>
($\Rightarrow$ they constitute a <u>finite</u> set, that hence can be counted)

The sol. space to the ODE $\dot\gamma(t) + \nabla_{\gamma(t)} f = 0$ with limits $\gamma(-\infty) = p$, $\gamma(+\infty) = q$ is

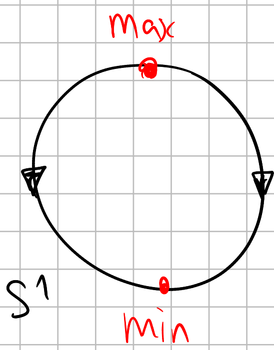
- of expected dimension $\mathrm{index}(p) - \mathrm{index}(q) - 1$ (modulo reparam. $\gamma(t) \mapsto \gamma(t + t_0)$)
- <u>regular</u> if $\gamma \mapsto \dot\gamma + \nabla_\gamma f$ has zero as a regular value

(a sol. is <u>rigid</u> if both properties hold)

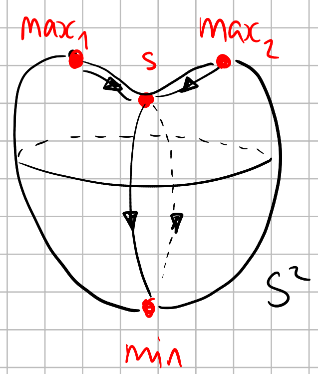


solution to $F(x) = 0$ is <u>not</u> regular (but exp. dim $= 0$)

Thm [Sard-Smale] For a generic Riemannian metric g on N, any flow-line of expected dimension zero ($\Leftrightarrow$ limits two crit pts of consecutive indices) is regular.



$$\partial(\max) = (\pm 1 \pm 1)\cdot \min$$
$$= 0 \quad \text{over } \mathbb{Z}_2$$
(in fact $=0$ also over $\mathbb{Z}$)



$$\partial(\max_i) = \pm s$$
$$\partial s = \partial^2(\max_i) = 0$$

Thm [Morse-Smale-Witten] • $\partial^2 = 0$ (for generic metric)

- The Morse homology $H_i^{Morse}(M) := H(C_\bullet^{Morse}(f), \partial)$ is invariant under
  - choice of Morse function $f: M \to \mathbb{R}$
  - choice of generic Riemannian metric g on M
  - homotopy equivalence $M \simeq M'$ of manifolds

  (the reason is that Morse homology computes singular homology / cellular homology)

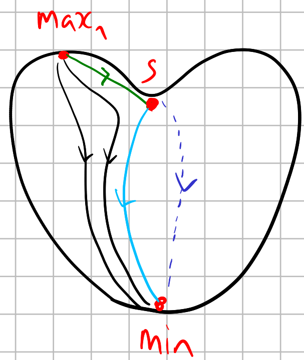
Cor If M has "large homology groups" (a topological invariant), then $df = 0$ must have many solutions for $f: M \to \mathbb{R}$ Morse.

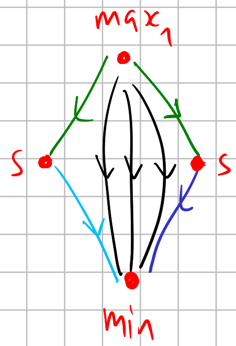
(More useful if f is a functional, e.g. path-length, and solutions to $df = 0$ are solutions to some ODE/PDE, e.g. the geodesic equation)

Illustration of $\partial^2 = 0$

$\partial^2(max_1) =$

$(= \partial(s)) = 0$





Exercise 28 Show that if M is connected, then $H_0^{Morse}(M;\mathbb{Z}_2) = \mathbb{Z}_2$ (Hint: Use the fact that M admits a Morse function with a unique minimum)

Conclude that $H_0^{Morse}(M;\mathbb{Z}_2) = \mathbb{Z}_2^{\pi_0(M)}$ in general

Assume M^n is compact (without boundary) $\Rightarrow$ $-f$ Morse & bdd. from below

"Poincare duality":

$$p \in Crit(f),\ \text{index} = i \quad \overset{bij}{\longleftrightarrow} \quad p \in Crit(-f),\ \text{index} = \dim M - i = n-i$$

$$\text{max} \quad \longleftrightarrow \quad \text{min}$$

$$(\partial^f)^* \quad = \quad \partial^{(-f)} \qquad / \ \mathbb{Z}_2 \text{ coeff.}$$

Exercise 29 Use the above Poincare duality together with Exc. 28 to show that $H_n^{Morse}(M;\mathbb{Z}_2) = \mathbb{Z}_2$ for M closed & connected

(In fact: $H_i^{Morse}(M;\mathbb{Z}_2) \cong H_{n-i}^{Morse}(M;\mathbb{Z}_2)$ for all i)

<u>Conclusion</u> $H_i(M^n;\mathbb{Z}_2) = \begin{cases} 0 & i>0 \\ \mathbb{Z}_2 & i=n \\ \vdots & \vdots \\ \mathbb{Z}_2 & i=0 \\ 0 & i<0 \end{cases}$ if M compact, connected and closed n-dim. mfd

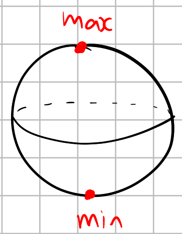
(same is true over $\mathbb{Z}$ if M is orientable)

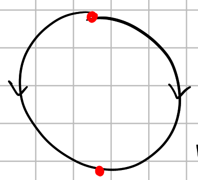
Ex

- $H_i(\mathbb{R}^n) = \begin{cases} \mathbb{Z}, & i=0 \\ 0, & o.w. \end{cases}$ use the Morse function $\|x\|^2$ which is proper & bounded from below

- $H_i(S^n) = \begin{cases} \mathbb{Z}, & i=0,n, \\ 0 & o.w. \end{cases}$ no rigid gradient flow lines for the std. Morse function when $n>1$

$n>1$





$n=1$

non-trivial computation unless $\mathbb{k}=\mathbb{Z}_2$

$\partial max = \pm 2\, min$ or $\boxed{\partial max = 0 \cdot min}$?

true answer