Recall

- $$H_i^{\mathrm{Morse}}(\mathbb{R}^n) = \begin{cases} \mathbb{Z}, & i=0 \\ 0, & \text{o.w.} \end{cases}$$

- $$H_i^{\mathrm{Morse}}(S^n) = \begin{cases} \mathbb{Z}, & i=0,n, \\ 0 & \text{o.w.} \end{cases}$$

Exercise 29 Compute $H_i^{\mathrm{Morse}}(\Sigma_g; \mathbb{Z}_2)$.

Fact $H_1^{\mathrm{Morse}}(M) = \pi_1(M)/\mathrm{comm}$ for general connected M.

DeRham Cohomology

An alternative complex that computes the homology of the dual co-complex $\left(\left(C_\cdot^{Morse}(M)\otimes\mathbb{R}\right)^*, d=\partial^*\right)$ of the Morse complex.

$$0 \to C^0_{DR}(M) \xrightarrow{d} C^1_{DR}(M) \xrightarrow{d} \dots \to C^n_{DR}(M) \to 0 \to \dots$$

infinite dimensional $\mathbb{R}$-vector spaces

$\eta^i \in C^i_{DR}(M)$ differential i-form

The pairing with $p \in C_i^{Morse}(M)$, $p \in Crit(f)$ index$(p)=i$, is described as follows:

$W^u(p)$: unstable (open) manifold points $\subseteq M$ that limit to $p \in Crit(f)$ under the positive gradient flow.

Exercise 30 Use the Morse lemma to find a smooth parametrisation of $W^u(p)$ by $\Phi_p : \mathbb{R}^{\text{index } p} \hookrightarrow M$. You are allowed to choose the metric freely near p to simplify the problem.

Pairing is given by:

$$\langle \eta^i, p \rangle := \int_{\Phi_p} \eta^i \in \mathbb{R} \rightsquigarrow \text{an } \mathbb{R}\text{-linear functional on } C_i^{Morse}(f)$$

$\uparrow$ index $= i$

More generally, the differential i-forms $C^i_{DR}(M)$ can be integrated over i-dim smooth simplices $\gamma : \Delta^i \xrightarrow[C^\infty]{} M$ s.t. moreover

$$\boxed{\text{The integral } \int_\gamma \eta^i \text{ is invariant under orientation-preserving reparametrisation}}$$

(In particular, η^i is not simply a measure on M, but a quantity that satisfies $(\gamma \circ \varphi)^* \eta = \varphi^*(\gamma^* \eta) = \det\left(\frac{\partial \varphi^i}{\partial x^j}\right) \cdot \gamma^* \eta$ for $\varphi : \Delta^i \to \Delta^i$ reparametrisation)

<u>Concretely</u>:

$C^0_{DR}(M) := C^\infty(M, \mathbb{R})$ $\qquad \Delta^0 = \{pt\}$ $\qquad \int_{\pm pt} g = \pm g(pt)$

$C^1_{DR}(M) :=$ smooth sections of the cotangent bundle $\underbrace{T^* M}_{\text{fibre-wise dual of } TM} \twoheadrightarrow M$

i.e. $V_{pt} \in T_{pt} M$: $\eta^1(V_{pt}) \in \mathbb{R}$ if $\eta^1 \in C^1_{DR}(M)$

Recall $f \in C^\infty(M, \mathbb{R})$, $df(V_{pt}) =$ differential of f along $V_{pt} \in T_{pt} M$
$\Rightarrow$ $df \in C^1_{DR}(M, \mathbb{R})$

In local coordinates: $T\mathbb{R}^n = \mathbb{R}^n_{\underline{x}} \times \mathbb{R}^n_{\underline{y}}$ $\quad T_{\underline{x}}\mathbb{R}^n = \langle \partial x_1, \dots, \partial x_n \rangle$

$$T^*\mathbb{R}^n = \mathbb{R}^n_{\underline{x}} \times \mathbb{R}^n_{\underline{y}} \quad T^*_{\underline{x}}\mathbb{R}^n = \langle dx_1, \dots, dx_n \rangle$$

$$dx_i(\partial_{x_j}) = \delta_{ij}$$

$$\Delta^1 = [0,1] \quad \int_\gamma \eta^1 := \int_0^1 \eta\left(\frac{d\gamma}{dt}\right) dt$$

change of variable formula for integration

$$\left[\begin{array}{l}\text{s(t) orientation pres.}\\ \text{change of coord}\end{array}\right] = \int_0^1 \eta\left(\frac{ds}{dt}\cdot\frac{d\gamma}{ds}\right)\cdot dt \overset{\text{(lin.)}}{=} \int_0^1 \eta\left(\frac{d\gamma}{ds}\right)\cdot\frac{ds}{dt}\cdot dt \overset{\downarrow}{=} \int_a^b \eta\left(\frac{d\gamma}{ds}\right)\cdot ds$$

<u>Stokes theorem</u>

(fundamental thm. of calculus)

$$\int_\gamma df = \int_0^1 df(\dot\gamma)\,dt \underset{\text{fund thm}}{=} f(\gamma(1)) - f(\gamma(0)) = \int_{\partial\gamma} f = \int_{\gamma(1)-\gamma(0)} f$$

$\partial\gamma$ ← boundary of the 1-simplex γ

i.e. $\langle df, \gamma\rangle = \langle f, \partial\gamma\rangle$

$C^i_{DR}(M)$ = smooth sections of the fibre-wise exterior algebra

$$\underbrace{T^*_{pt}M \wedge \dots \wedge T^*_{pt}M}_{i \text{ factors}} \subseteq T^*_{pt}M \otimes \dots \otimes T^*_{pt}M \quad \left(\Rightarrow C^i_{DR}(M) = 0 \quad i > \dim M\right)$$

i.e.: the fibre over $pt \in M$ consists of <u>anti-symmetric</u> i-multi linear maps from $T_{pt}M \otimes \dots \otimes T_{pt}M$ to $\mathbb{R}$.

<u>Good behaviour</u> $\eta^n(A(v_1), \dots, A(v_n)) = \det A \cdot \eta^n(v_1, \dots, v_n)$ (c.f. Jacobian factor in change of var. rule)

$d: C^i_{DR}(M) \to C^{i+1}_{DR}(M)$ is a derivation of the graded-commutative algebra $\left(\bigoplus_{i=0}^n C^i_{DR}(M), \wedge\right)$ determined by $d: C^0 \to C^1$ & the "graded Leibniz rule": $\boxed{d(\alpha^i \wedge \beta^j) = d(\alpha^i)\wedge\beta^j + (-1)^i \alpha^i \wedge d(\beta^j)}$

Thm (DeRham) • $d^2=0$ (closed i-forms modulo exact i-forms)

• $H^i_{DR}(M) := \ker d|_{C^i_{DR}(M)} / d(C^{i-1}_{DR}(M))$ is invariant under homotopy equiv.

• $H^i_{DR}(M) = H^{Morse}_i(M;\mathbb{R})^*$

Cor • $H^i_{DR}(\mathbb{R}^n) = H^i_{DR}(\mathbb{R}^0) = \begin{cases} \mathbb{R} & i=0 \\ 0 & i\neq 0 \end{cases}$ (Poincaré lemma)

• $H^i_{DR}(S^n) = \left(H^{Morse}_i(S^n;\mathbb{R})\right)^* = \begin{cases} \mathbb{R} & i=0,n \\ 0 & i\neq 0,n \end{cases}$

Exercise 31 Compute $H^1_{DR}(S^1)$ by hand.

Calculate $H^i_{DR}(M)$ from a Morse function

$\overset{\text{closed}}{U} \subseteq M$ domain w. smooth boundary (e.g. $U = M_{\leq c} \subseteq M$, c generic)

$C^i_{DR}(M,U) := \{\eta^i \text{ s.t. } \eta^i|_U \equiv 0\}$ relative DeRham complex

$H^i_{DR}(M,U) := \ker d|_{C^i_{DR}(M,U)} / d(C^{i-1}_{DR}(M,U))$ relative DeRham cohomology

Fact of homological algebra

A short exact sequence of $\mathbb{R}$-complexes

$$0 \to C^\bullet_{DR}(M,U) \xhookrightarrow{\text{ind.}} C^\bullet_{DR}(M) \xrightarrow{\text{quot}} C^\bullet_{DR}(M)/C^\bullet_{DR}(M,U) \to 0$$

$$C^\bullet_{DR}(M) \xrightarrow{\text{restr.}} C^\bullet_{DR}(U) \cong C^\bullet_{DR}(M)/C^\bullet_{DR}(M,U)$$

induces a long exact sequence

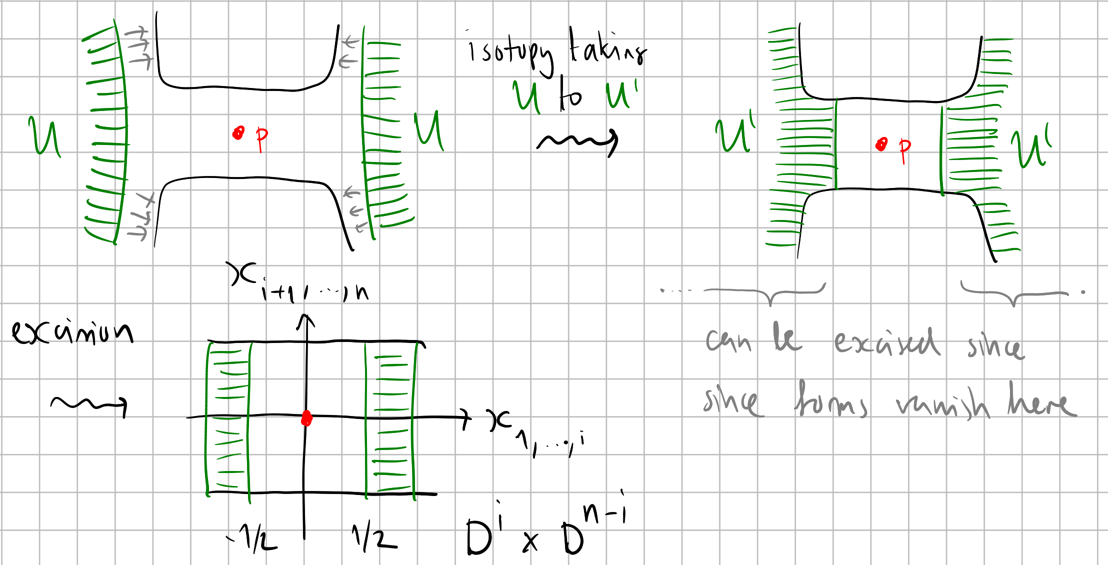
$$\boxed{\cdots \to H^{i-1}_{DR}(U) \to H^i_{DR}(M,U) \xrightarrow{[\text{incl.}]} H^i_{DR}(M) \xrightarrow{[\text{restr.}]} H^i_{DR}(U) \to H^{i+1}_{DR}(M,U) \to \cdots}$$

via the "snake lemma".

Exercise 32

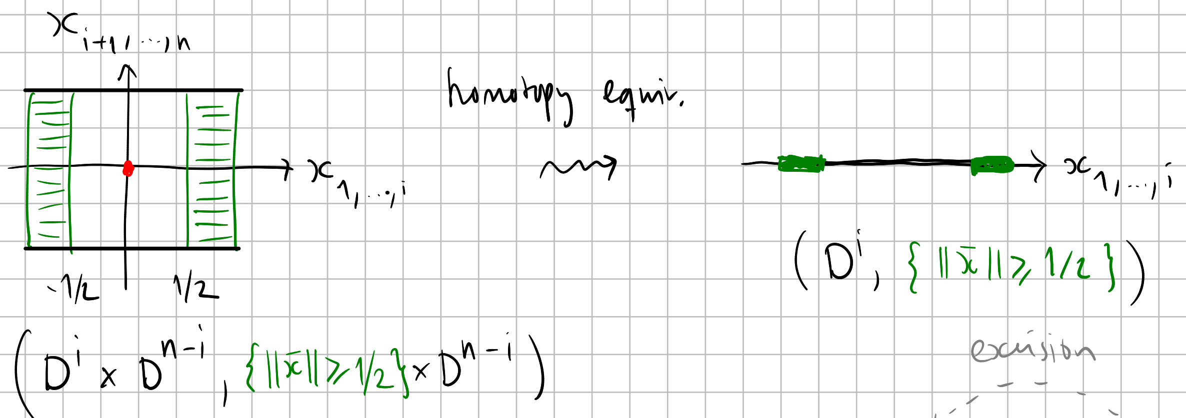
Use $H^i_{DR}(\mathcal{U}) \cong H^i_{DR}(\mathcal{U}\setminus\partial\mathcal{U})$ & the LES above to show that
$$H^i_{DR}\left(S^n, \{\|\bar{x}\|^2=1 \mid x_{n+1}\leq 0\}\right) = \begin{cases} \mathbb{R} & i=n \\ 0 & i\neq n\end{cases}$$

We apply the LES to $C^\bullet_{DR}(\overbrace{M_{\leq f(p)+\varepsilon}}^{M}, \overbrace{M_{\leq f(p)-\varepsilon}}^{\mathcal{U}})$ where $f^{-1}(p)$ contains a unique Morse-type critical point of index $= i$



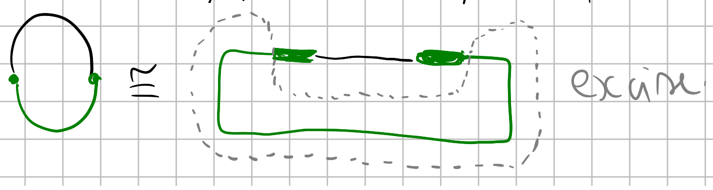
$H^\bullet_{DR}(M_{\leq f(p)+\varepsilon}, \mathcal{U}) = H^\bullet_{DR}(M_{\leq f(p)+\varepsilon}, \mathcal{U}')$ (by htpy. invariance)

$$\Rightarrow \quad H^\bullet_{DR}(M_{\leq f(p)+\varepsilon}, M_{\leq f(p)-\varepsilon}) \cong H^\bullet_{DR}\left(D^i\times D^{n-i}, \{\|\bar{x}\|\geq 1/2\}\times D^{n-i}\right)$$



excision

$$\Rightarrow \quad H^\bullet_{DR}(M_{\leq f(p)+\varepsilon}, M_{\leq f(p)-\varepsilon}) \cong H^\bullet_{DR}(D^i, \{\|\bar{x}\|\geq 1/2\}) = H^\bullet_{DR}(S^i, \{x_{n+1}\leq 0\})$$



$$\underset{[\text{Exc. }32]}{=} \begin{cases}\mathbb{R}, & \bullet = n \\ 0, & \text{o.w.}\end{cases}$$