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Computing $H^i_{DR}(M)$ from a Morse function

$f: M^n \to \mathbb{R}$ Morse function which is proper & bounded from below with distinct values of each critical point.

Critical values are

$$f(p_1) < f(p_2) < f(p_3) < \dots$$

$\uparrow$ global min.

Restriction of i-forms $r_i^{a,b}: C^i_{DR}(M_{\leq b}) \to C^i_{DR}(M_{\leq a})$ for regular values $b > a$ are natural chain maps (the differential in the target is the restriction of the differential)

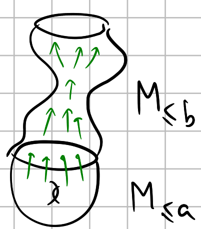
$$\Rightarrow \boxed{\text{Restriction morphism } r_i^{a,b}: H^i_{DR}(M_{\leq b}) \to H^i_{DR}(M_{\leq a})}$$

Goal: Understand the restriction maps $r^{a,b}$.

Lem 1.) If $[a,b] \cap \mathrm{Crit}(p) = \emptyset$, then $r_i^{a,b}: H^i_{DR}(M_{\leq b}) \xrightarrow{\cong} H^i_{DR}(M_{\leq a})$ is an isomorphism, moreover $\exists$ a diffeomorphism $M_{\leq a} \cong M_{\leq b}$.

diffeomorphism induced by isotopy

$$\begin{tikzcd} H^i_{DR}(M_{\leq b}) \arrow[r, "r_i^{a,b}", "\cong"'] \arrow[dr, "\mathrm{id}"'] & H^i_{DR}(M_{\leq a}) \arrow[d, "\cong \text{ induced by diffeom.}"] \\ & H^i_{DR}(M_{\leq b}) \end{tikzcd}$$



2.) In general $r_i^{a,b} = r_i^{p_j} \circ r_i^{p_{j+1}} \circ \dots \circ r_i^{p_{j+k}}$

where $r_i^p = r_i^{f(p)-\varepsilon, f(p)+\varepsilon}$ (after appropriate isomorphisms from (1).)

The previous lecture we obtained the following LES:

For each $p \in \mathrm{Crit}(f)$, $i_0 := \mathrm{index}(p) \in \{0, 1, \dots, n = \dim M\}$

restriction of forms

"connecting homomorphism" from the Snake Lemma

inclusion of forms

$$\boxed{\begin{array}{l} \dots \xrightarrow{\delta^p_{i-1}} \underbrace{H^i_{DR}(M_{\le f(p)+\varepsilon}, M_{\le f(p)-\varepsilon})}_{= \begin{cases} \mathbb{R}\cdot\alpha_p^{i_0} & i = i_0 \\ 0 & \text{o/w} \end{cases}} \xrightarrow{\iota^p_i} H^i_{DR}(M_{\le f(p)+\varepsilon}) \xrightarrow{r^p_i} H^i_{DR}(M_{\le f(p)-\varepsilon}) \xrightarrow{\delta^p_i} \\ \qquad H^{i+1}_{DR}(M_{\le f(p)+\varepsilon}, M_{\le f(p)-\varepsilon}) \to \dots \end{array}}$$

Hence: $r^p_i : H^i_{DR}(M_{\le f(p)+\varepsilon}) \xrightarrow{\cong} H^i_{DR}(M_{\le f(p)-\varepsilon})$

$\qquad i \ne i_0 - 1 \Rightarrow \delta^p_i = 0)$

$\boxed{i \ne i_0 \Rightarrow \iota^p_i = 0 \Rightarrow r^p_i \text{ injective}}$

$\boxed{i \ne i_0 - 1 \Rightarrow \delta^p_{i-1} = 0 \Rightarrow r^p_{i-1} \text{ surjective}}$

$\Rightarrow$ $\boxed{\begin{array}{l} r^p_i \text{ isomorphism whenever} \\ i \ne i_0 \ne i+1 \\ \quad\; i_0 = \text{index } p \end{array}}$

There are two possibilities for $\delta^p_{i_0-1}$ & $\iota^p_{i_0-1}$

(1) $\quad \delta^p_{i_0-1} : H^{i_0-1}_{DR}(M_{\le f(p)-\varepsilon}) \twoheadrightarrow H^{i_0}_{DR}(M_{\le f(p)+\varepsilon}, M_{\le f(p)-\varepsilon}) = \mathbb{R}$ surjective ($\Leftrightarrow \iota^p_{i_0} = 0$)

or

(2) $\quad \iota^p_{i_0} : H^{i_0}_{DR}(M_{\le f(p)+\varepsilon}, M_{\le f(p)-\varepsilon}) = \mathbb{R} \hookrightarrow H^{i_0}_{DR}(M_{\le f(p)+\varepsilon})$ injective ($\Leftrightarrow \delta^p_{i_0-1} = 0$)

Case (B): $H^{i_0}_{DR}(M_{\le f(p)+\varepsilon}) \overset{r^p_{i_0}}{\twoheadrightarrow} H^{i_0}_{DR}(M_{\le f(p)-\varepsilon})$ with one-dim kernel $= \mathrm{im}\, \delta^p_{i_0}$
$\Leftrightarrow$ (2)

In fact: the class $\alpha^p_{i_0}$ is born

("a cohomology class is born in degree i_0 as we pass from $f = f(p) - \varepsilon$ to $f(p) + \varepsilon$")

Case (D): $H^{i_0-1}_{DR}(M_{\le f(p)+\varepsilon}) \overset{r^p_{i_0-1}}{\hookrightarrow} H^{i_0-1}_{DR}(M_{\le f(p)-\varepsilon})$ with one-dim cokernel $=$
$\Leftrightarrow$ (1)

$$= H^{i_0-1}_{DR}(M_{\le f(p)-\varepsilon}) / \mathrm{im}(r^p_{i_0-1}) \xrightarrow[\delta_{i_0-1}]{\cong} H^{i_0}_{DR}(M_{\le f(p)+\varepsilon}, M_{\le f(p)-\varepsilon})$$

("a cohomology class dies in degree $i_0 - 1$ as we pass from $f = f(p) - \varepsilon$ to $f(p) + \varepsilon$")

slightly misleading: a codim 1 subspace survives is more accurate.

Fact • Case (D) occurs precisely when:

• $\partial^f(p) \neq 0$ ($\mathbb{R}$-coefficients), or equivalently

$\Leftrightarrow$ • $\exists\, [\eta^{i_0-1}] \in H^{i_0-1}_{DR}(M_{\leq f(p)-\varepsilon})$ s.t. $\displaystyle\int_{\Phi_p(S^{i_0-1})} \eta^{i_0-1}$

This is a consequence of $H^\bullet_{DR}(M) \cong \left(H^{Morse}_\bullet(M;\mathbb{R})\right)^*$

• $H^0_{DR}(M) \cong$ {locally constant functions} $\cong \mathbb{R}^{|\pi_0(M)|}$

• M^n closed ($\partial M = \emptyset$) and connected n-dim manifold

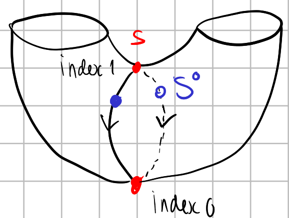
choice of volume form

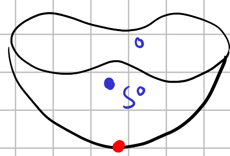
$H^n_{DR}(M) \ni \eta^n \mapsto \int_M \eta \in \mathbb{R}$ is an isomorphism, $H^n_{DR}(M) = \mathbb{R}\cdot \eta^n_{vol}$

in local coordinates: $\eta^n = \underbrace{\rho(x)}_{\geq 0}\, dx_1 \wedge \dots \wedge dx_n$. supported e.g. in loc. chart.

Ex Case (B):

$M_{\leq f(s)+\varepsilon} \cong S^1 \times I \sim_{ht} S^1$



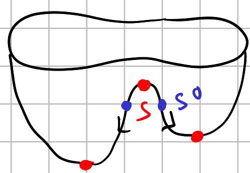


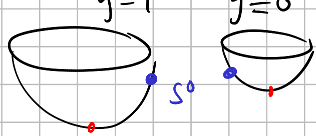
$\int_{S^0} g = 0$ (g loc. cst)

$M_{\leq f(s)-\varepsilon} \cong D^2$

$$\begin{array}{ccc} H^1_{DR}(S^1\times I) & \twoheadrightarrow & H^1_{DR}(D^2) \\ \wr\!\parallel & & \parallel \\ \mathbb{R}\cdot d\theta & & 0 \end{array}$$

Case (D):



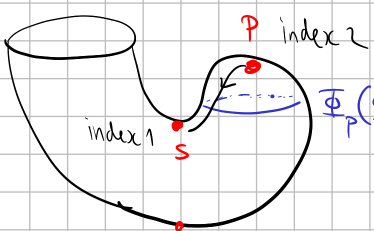


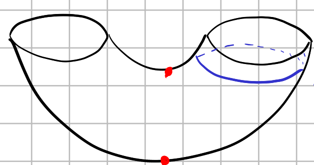
$\int_{S^0} g = 1 \neq 0$

$$\begin{array}{ccc} H^0(D^2) & \hookrightarrow & H^1(D^2 \sqcup D^2) \\ \wr\!\parallel & & \wr\!\parallel \\ \mathbb{R} & \xrightarrow{\cong} \mathbb{R}\cdot(1,1) \subseteq & \mathbb{R}^2 \end{array}$$

Case (D):

$M_{\leq f(p)+\varepsilon} \cong D^2$





$M_{\leq f(p)-\varepsilon} \cong S^1 \times I \sim_{ht} S^1$

$\int_{\Phi_p(S^1)} d\theta \neq 0$

$$\begin{array}{ccccc} H^1(D^2) & \xhookrightarrow{r_1^p} & H^1(S^1) & \xrightarrow{\cong} & H^1(M_{\leq f(p)+\varepsilon}, M_{\leq f(p)-\varepsilon}) \\ \parallel & & \parallel & & \parallel \\ 0 & & \mathbb{R} & & \mathbb{R} \end{array}$$

$$H^i_{DR}\left(M_{\leqslant f(p)+\varepsilon}, M_{\leqslant f(p)-\varepsilon}\right) = \begin{cases} \mathbb{R}\cdot\alpha_p^{i_0} & i=i_0 \\ 0 & o/w \end{cases} \xrightarrow{\iota_i} H^i_{DR}\left(M_{\leqslant f(p)+\varepsilon}\right) \quad (\text{possibly: } \iota_{i_0}=0)$$

Def [Viterbo] The spectral invariant of a class $\eta\in H^i_{DR}(M_{\leqslant b})$ is

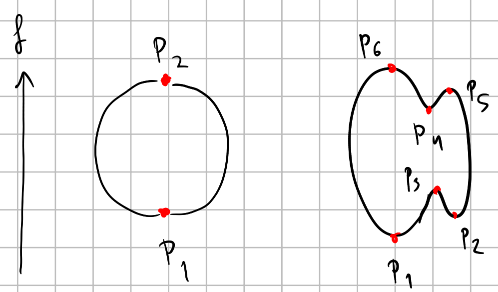
$$c^f(\eta) := \inf\left\{a\in\mathbb{R} \mid r^{a,b}(\eta)\neq 0\right\}\in\mathbb{R}$$

Exercise 33 Show that all spectral values are critical points of f at which Case (B) (class born) occurs.

Exercise 34 Show that there are $\dim H^i_{DR}(M_{\leqslant b})$ number of spectral values in $(-\infty, b)$. (Hint: use LES & argue by induction)

Exercise 35 Recall that $H^i_{DR}(S^1) = \mathbb{R}\eta^1\oplus\mathbb{R}\eta^0$, $\eta^1 = d\theta$, $\eta^0 = \mathrm{cst}_1$

compute $c^f(\eta^i)$ for



The Barcode (Topological data analysis)

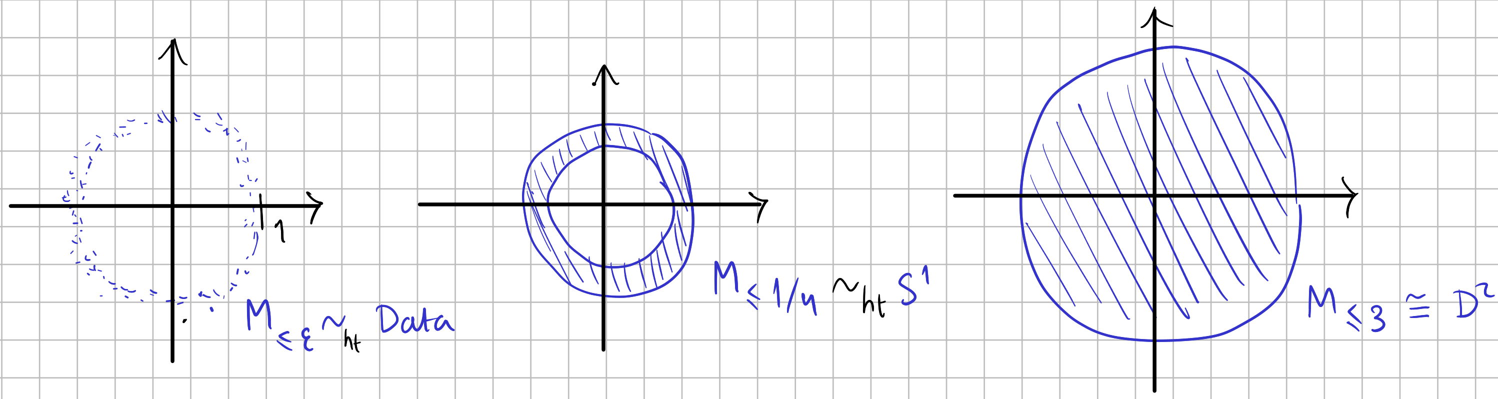
Introduced by Carlson et al. in the field of topological data analysis ?

Typical setting: $M = \mathbb{R}^N$, $\text{Data} \subseteq \mathbb{R}^N$ finite set of points

$$f : \mathbb{R}^N \longrightarrow \mathbb{R}$$
$$\bar{x} \longmapsto \text{dist}(\bar{x}, \text{Data})^2$$

(perturbed to Morse fcn w. distinct crit. values)

$M_{\leq \varepsilon} \sim \text{Data}$, $M_{\leq R} \cong D^N$, $R \gg 0$, neither of these sublevels have interesting topology, but for intermediate levels $M_{\leq t}$, many things can happen...



Q: The topology of $M_{\leq t}$ changes as $t \in \mathbb{R}$ varies. Which cohomology classes <u>persist</u>, and for how long? (Applicable to gen. M)

Def The <u>Barcode of (M, f)</u> is a union of intervals $[s, e)$ with $s \in \mathbb{R}$, $e \in (s, +\infty]$ determined by:

<u>Starting points of bars</u>: Are in bijection with $f(p) \in \mathbb{R}$, $p \in \text{Crit}(f)$, at which Case (B) (class born) occurs

<u>End points of bars</u> ($\neq +\infty$) Are in bijection with $f(q) \in \mathbb{R}$, $q \in \text{Crit}(f)$ at which Case (D) (class dies) occurs.

The bar that ends at $f(q)$ has starting point given by the spectral value for $M_{\leq f(q) - \varepsilon}$ that disappears for $M_{\leq f(q) + \varepsilon}$

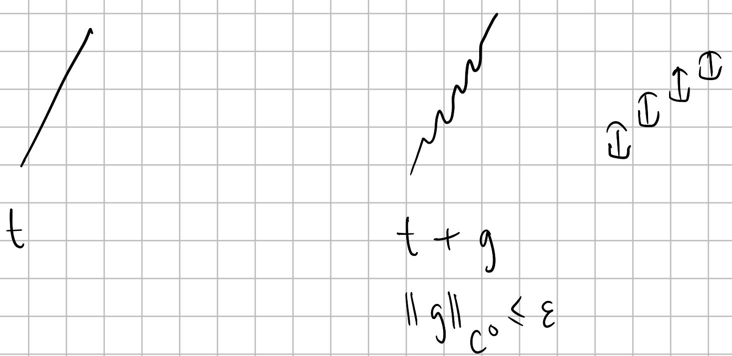
The last condition makes sense because of Exercise 33 & 34.

Rmk Exc. 34 $\Rightarrow$ $\dim H_{DR}(M_{\leq b}) = \#$ bars that intersect level b.
(also works for singular sublevel sets)

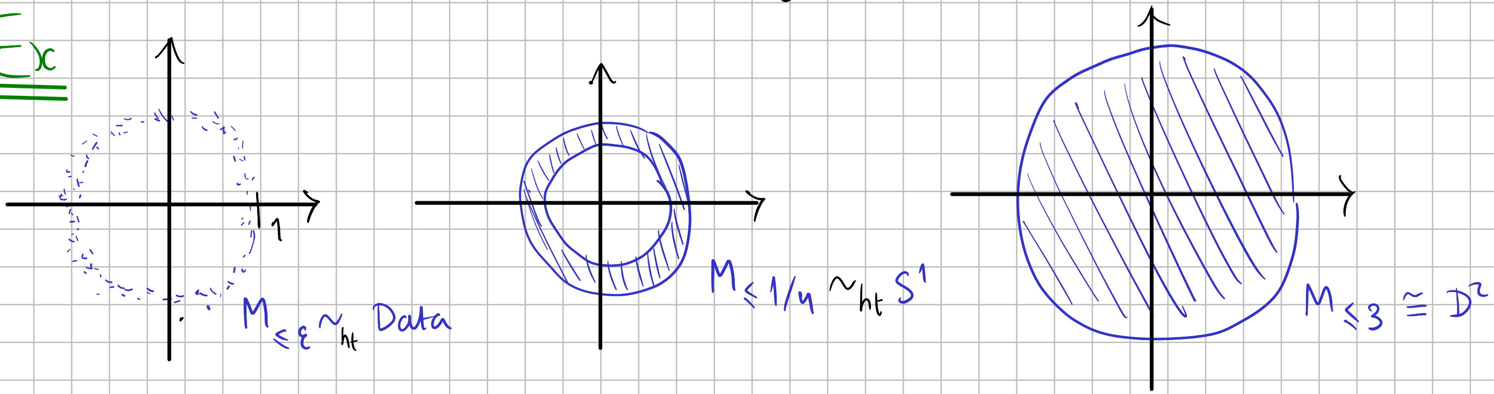
Exercise 36 Compute the barcodes for the different f in Exc. 35.

Main feature: Up to removal/addition of short bars &
a perturbation of the endpoints of bars ("bottleneck distance")
the barcode depends continuously only on the C^0 behaviour of f

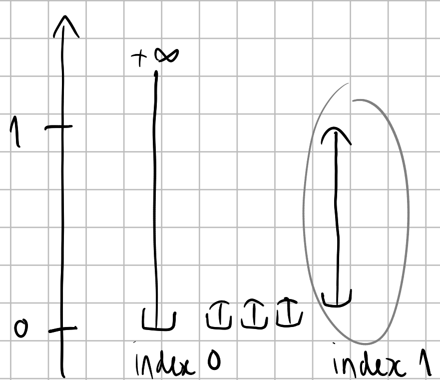
Consequence: The barcode has properties that are stable under
perturbations of the data and the function f.



Ex



Barcode:



$\Rightarrow$ data has the shape of S^1.