 georgios.dimitroglou@math.uu.se

§ III Knot theory

In knot theory we study the question of the classification of subsets

$$K \subseteq M^n$$

up to a relation such as

- Diffeomorphism of M, or even stronger:
- Smooth isotopy of M

$\Phi : M \times [0,1]_t \to M$ smooth

$\Phi_t := \Phi|_{M \times \{t\}}$ diffeomorphism $\forall t$

$\Phi_0 = \mathrm{id}_M$

Fact
- A smooth isotopy is determined by $\underbrace{V_t(p)}_{\text{infinitesimal generator}} = \dot{\Phi}_t(p) = \frac{\partial}{\partial t}\Phi_t(p) \in T_{\Phi_t(p)}M$

  Reason: Unicity of solutions to ODE: $\begin{cases} \Phi_0(p) = p \\ \dot{\Phi}_t(p) = V_t(\Phi_t(p)) \end{cases}$

- A diffeomorphism is isotopic to id_M $\iff$ it lives in the same component $\pi_0(\mathrm{Diff}^\infty(M))$ as id_M, where the topology in $\mathrm{Diff}^\infty(M)$ is the ∞-dim. Frechét Lie group with topology in which convergence = uniform C^k-convergence for all $k \geq 0$ (For simplicity: M compact)

- $\Phi_t \in \mathrm{Diff}^\infty(M)$ is a smooth path, $\frac{\partial}{\partial t}\Phi_t = V_t \in T_{\Phi_t}\mathrm{Diff}^\infty(M)$

$\pi_0(\mathrm{Diff}^\infty(S^4))$ is not known

Also $\pi_0(\{\varphi \in \mathrm{Diff}^\infty(\mathbb{R}^4) \mid \varphi|_{\mathbb{R}^4 \setminus B^4} = \mathrm{id}_{\mathbb{R}^4}\})$ is unknown.

Thm $\pi_0(\mathrm{Diff}^\infty(S^n)) = \{[id], [\text{reflection}]\}$ $\quad n=2$ (Smale '58)

$n=3$ (Cerf '64)

Not true for $n=6, \ldots$ $\quad |\pi_0(\mathrm{Diff}^\infty(S^6))| = 2\cdot 28$ (Milnor, Cerf, Smale)

Typically we are interested in the case when $K^k \subseteq M^n$ is a submanifold, by which we mean

Def $K \subseteq M$ is a submanifold if each $p \in K$ has a nbhd $U \overset{\text{open}}{\subseteq} M$ that admits a local chart $\varphi: U \to \mathbb{R}^n$ in which $\varphi(K) = \mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^n$

Implicit function thm: K is a submanifold $\Leftrightarrow$ K is locally of the form $K \cap U = f^{-1}(0)$, where $f: U \xrightarrow{C^\infty} \mathbb{R}^{n-k}$ $\quad$ ($n-k$ = codimension of K)

0 regular value

Rmk
- A submanifold of $\mathbb{R}^N$ is the same as a manifold
- Any submanifold is a manifold, and the inclusion map is a smooth injective immersion, i.e. the differential is everywhere injective

($T_p\gamma : T_pK \to T_{\gamma(p)}M$ injective for all p)

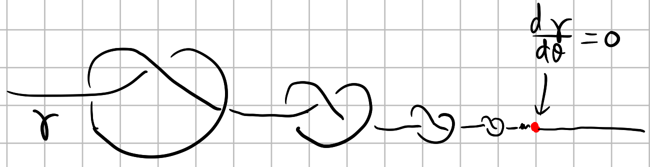
Conversely, a useful way to see that a subset is a submanifold is to verify that it is the image under such a map;

Thm If $\gamma: K \hookrightarrow M$ is a proper (e.g. K compact) smooth injective immersion, then $\gamma(K) \subseteq M$ is a submanifold.

($T_p\gamma : T_pK \to T_{\gamma(p)}M$ injective for all p)

Here, we study one-dimensional closed knots $K^1 \subseteq S^3$, i.e. admitting a smooth parametrisation

$$\gamma : S^1 \amalg \dots \amalg S^1 \xhookrightarrow{C^\infty} S^3 \qquad d\gamma/d\theta \neq 0 \text{ everywhere}$$



A wild knot (not submanifold)

The existence of a smooth isotopy follows from a similar condition: a smooth family of parametrised submanifolds can be generated by an ambient smooth isotopy.

Thm (Isotopy extension thm) If $\varphi : K \times [0,1]_t \xrightarrow{C^\infty} M$ smoothly parametrised a family of submanifolds, more precisely: φ_t proper, injective, immersion, then there exists a smooth isotopy $\Phi_t : M \to M$ s.t. $\varphi_t = \Phi_t \circ \varphi_0$.

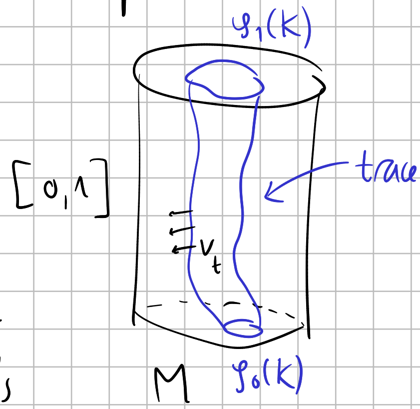
also called isotopy of submanifolds

$\frac{d}{d\theta}\varphi_t = 0$ or φ not smooth as a map from $K \times [0,1]$

Here the isotopy extension theorem does not apply (this is not an isotopy of knots)

Idea of proof Key point: Above thm implies that

$$K\times[0,1] \hookrightarrow M\times[0,1]$$
$$(p,t) \longmapsto (\varphi_t(p),t)$$



parametrises a submanifold: the trace of the family of submanifolds

$v_t(p) := \dot{\varphi}_t(p) \in T_{\varphi_t(p)}M$ smooth vector field, $(v_t(p),0) \in T_{(\varphi_t(p),t)}(M\times[0,1])$ smooth vector field defined along the trace

Goal: Construct a smooth extension of v_t to a globally defined vector field V_t, $V_t|_{\varphi_t(K)} = v_t$

- Local extensions exist by "definition of smoothness".
  $\Rightarrow \exists$ smooth local extension to ambient mfd.
- Use bump functions / tubular neighbourhood to patch together extension & make support contained to small nbhd of $\varphi_t(K)$

$\square$

Knot diagrams & Reidemeister moves

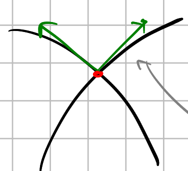
The isotopy question in S^3 is equivalent to that in $\mathbb{R}^3$:

Obviously: Any codim ≥ 1 knot $K \subseteq S^n$ can be perturbed (say by $A \in SO(4)$) so that it ends up inside $\mathbb{R}^n = S^n \setminus \{N\}$

Exercise 37 Use Sard's thm to show that two codim ≥ 2 submanifolds in $\mathbb{R}^n = S^n \setminus \{N\}$ are isotopic in S^n $\Leftrightarrow$ they are isotopic in $\mathbb{R}^n$.

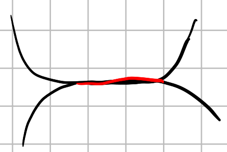
Thm For any smoothly parametrised knot $\gamma : K^1 \hookrightarrow \mathbb{R}^3$, for an $\underbrace{\text{open \& dense}}_{\text{"generic"}}$ choice of $A \in SO(3)$, the image under the orthogonal projection $pr : \mathbb{R}^3_{xyz} \to \mathbb{R}^2_{xy}$ satisfies the following:

$pr \circ A \circ \gamma$ is an
- immersed closed curve, with
- only <u>transverse</u> double points

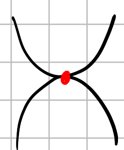


velocity vectors span a 2-dim plane

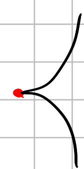
Ex For a generic rotation, we will thus <u>not</u> see a projection with:



non-discrete intersection

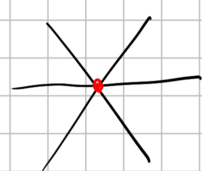


self-tangency



$y^2 = x^3$

cusp



triple point.

Idea of proof.

$$\frac{d\gamma}{d\theta} \Big/ \left\| \frac{d\gamma}{d\theta} \right\| : K \longrightarrow S^2 \quad \text{is smooth since} \quad \frac{d\gamma}{d\theta} \neq 0$$

Sard's thm: regular values (= complement of image) are open and dense. The complement of this image is precisely the directions along which the projection is immersed.

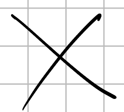
For double points one must analyse

$$\gamma \times \gamma : K \times K \setminus \overbrace{\{(\theta,\theta)\}}^{\text{"diagonal"}} \longrightarrow S^2$$
$$(\theta_1, \theta_2) \longmapsto \gamma(\theta_1) - \gamma(\theta_2) / \| \gamma(\theta_1) - \gamma(\theta_2) \| \qquad \square$$

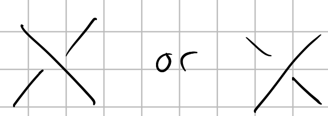
Ex The unknot: $\{0\} \times S^1 \subseteq \mathbb{R}^3$

Knot diagram: A closed immersed curve in $\mathbb{R}^2$ with

- transverse double points.



- additional data at each crossing: which arc is on top and which is below



Obviously: A smooth isotopy class of knot diagrams induces a well-defined isotopy class of knots (z-coordinate can be recovered up to smooth isotopy of knots).