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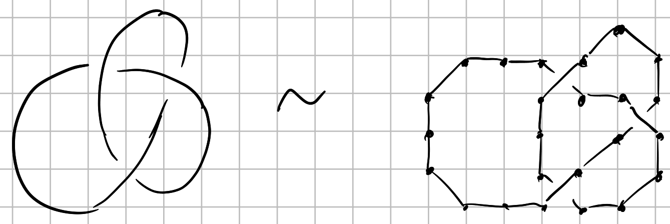
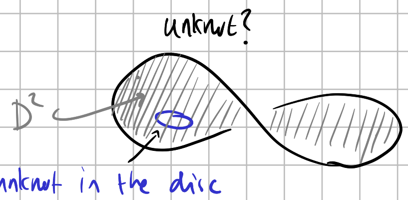
Knot diagrams in $\mathbb{R}^2$ / isotopy are easy to discretise:

E.g. 

Consequence: $K^1 \subseteq S^3$ up to smooth isotopy forms a $<$ countable set

(in fact: true for knots in all dimensions)



How to distinguish knots?

The unknot $K_0 := S^1 \times \{0\} \subseteq \mathbb{R}^3 = S^3 \setminus \{N\}$ can be characterised as

the unique knot that bounds a smooth embedding $D^2 \overset{\iota}{\hookrightarrow} S^3$

injective immersion ($T\iota$ injective)

Thm [Dehn, Papakyriakopoulos] $K \subseteq S^3$ is isotopic to the unknot if and only if $\pi_1(S^3 \setminus K) \cong \mathbb{Z}$.

Rmk $$S^3 \setminus K_0 = \Big\{ (\overbrace{x_1, y_1}^{(r_1,\theta_1)}, \overbrace{x_2, y_2}^{(r_2,\theta_2)}) \;\Big|\; \overbrace{x_1^2 + y_1^2 + x_2^2 + y_2^2}^{r_1^2+r_2^2} = 1,\ \overbrace{x_2^2 + y_2^2}^{r_2^2} > 0 \Big\}$$
$$= \{ r_1^2 < 1,\ r_2^2 = 1 - r_1^2 > 0 \} \cong D^2_{(r_1,\theta_1)} \times S^1_{\theta_2}$$

Alternatively: $$\begin{array}{ccccc} S^1 & \hookrightarrow & S^3 & \twoheadrightarrow & S^2 \\ & & \| & & \| \\ & & SU(2) & & \mathbb{C}P^1 \end{array}$$ Hopf fibration

$K_0 = S^1 \cdot (1,0) \in \mathbb{C}^2$ = fibre over $(1,0,0) \in S^2$ ($\leftrightarrow [1:0] \in \mathbb{C}P^1$)

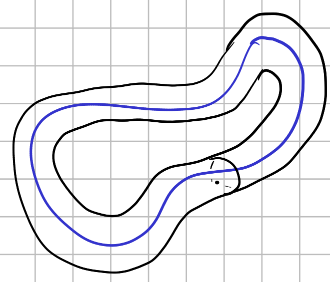
$\Rightarrow S^3 \setminus K_0 = S^1$-fibration over $S^2 \setminus \{pt\} \cong S^1 \times (S^2 \setminus \{pt\})$ (contractible base)

Exercise 38 Show that $\pi_1(\mathbb{R}^3 \setminus K) \xrightarrow{(incl)_*} \pi_1(S^3 \setminus K)$ is an isom for any K

$\pi_1(S^3\setminus K)$ actually detects the knot (is a complete knot invariant) if we add the data of the map $\pi_1(\partial(\nu K)) \to \pi_1(S^3\setminus K)$ induced by the inclusion

$$\pi_1(\partial(\nu K)) = \underbrace{\mathbb{Z}^2 \oplus \cdots \oplus \mathbb{Z}^2}_{|\pi_0(K)|}$$

$$\partial(\nu K) \subseteq S^3 \setminus K, \qquad \partial(\nu K) = T^2 \sqcup \cdots \sqcup T^2$$



<u>But</u> this invariant is not useful for practical purposes.

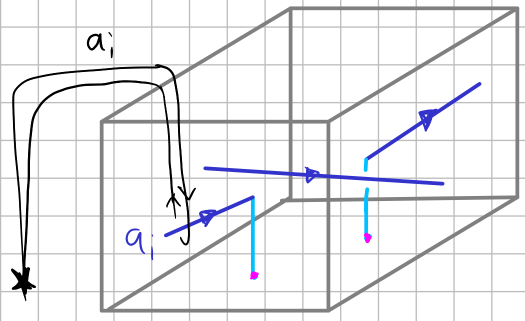
The Wirtinger Presentation of $\pi_1(S^3\setminus K)$

The <u>Knot group</u> (called the <u>link group</u> when $|\pi_0(K)|>1$) $\pi_1(S^3\setminus K)$ can be computed by using Seifert-van Kampen's Thm (Lecture 3)

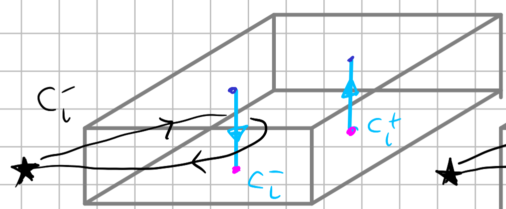
$$\begin{tikzcd}
\pi_1(U\cap V) \arrow[r, "(\mathrm{inc}_1)_*"] \arrow[d, "(\mathrm{inc}_2)_*"'] \arrow[dr, phantom, "\text{push-out}"] & \pi_1(U) \arrow[d, "(\mathrm{inc}_3)_*"] \\
\pi_1(V) \arrow[r, "(\mathrm{inc}_4)_*"'] & \pi_1(U\cup V) \cong \pi_1(U) \ast_{\pi_1(U\cap V)} \pi_1(V)
\end{tikzcd}$$

contains all arcs in the knot diagram (not the under-passes of the crossings)

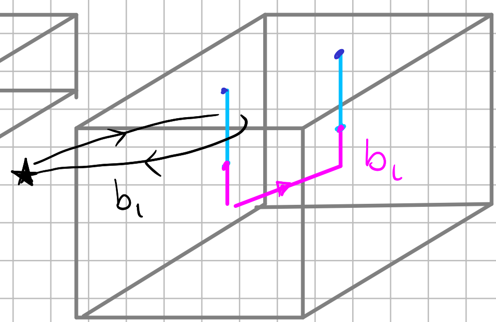
the under-passes of the crossings



$U = \mathbb{R}^2 \times (0,\infty) \setminus K$

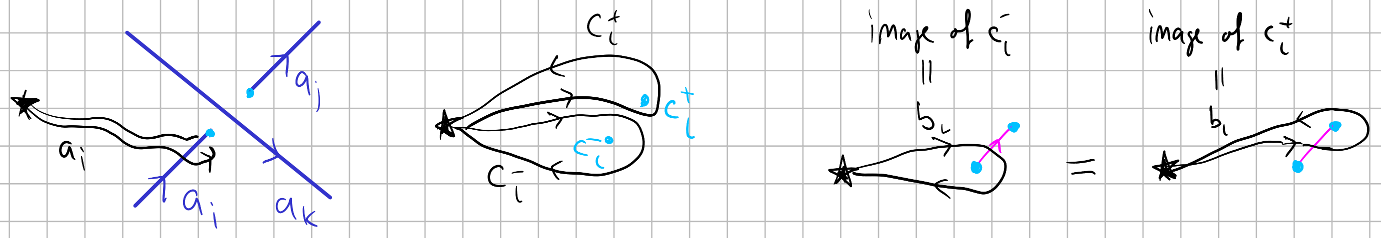


$U \cap V$



$V = \mathbb{R}^2 \times (-\infty, 1) \setminus K$

<u>Obs</u> all of $U, V, U\cap V \cong B^3 \setminus \{\text{unknotted arcs}\} \Rightarrow \pi_1$ freely generated by:

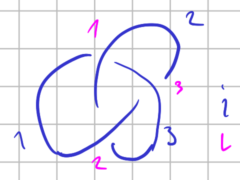


$\pi_1(U) = \langle a_j \rangle$ $\qquad \pi_1(U\cap V) = \langle c_\iota^-, c_\iota^+ \rangle, \quad \pi_1(V) = \langle b_\iota \rangle$

WLOG: agree away from crossing $\qquad$ image of c_ι^- & c_ι^+

j: enumerate arcs in the knot diagram

ι: enumerate crossings / under-arcs in the knot diagram

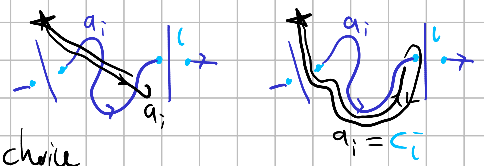


The maps $\pi_1(U\cap V) \underset{\eta_V}{\overset{\eta_U}{\rightrightarrows}} \begin{matrix}\pi_1(U)\\ \pi_1(V)\end{matrix}$ induced by the inclusion $U\cap V \hookrightarrow \begin{matrix}U\\V\end{matrix}$ are <u>surjective</u>

<u>w.l.o.g:</u> $b_\iota = \eta_V(c_\iota^-) = \eta_U(c_\iota^+)$

& $a_i = \eta_U(c_\iota^-)$

for appropriate choice

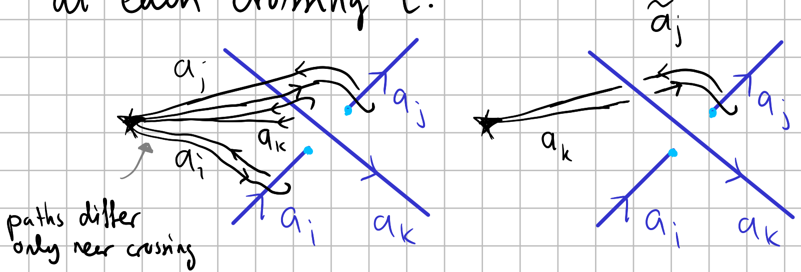


$\pi_1(U\cup V) = \langle a_j, b_\iota \mid r_\iota^+, r_\iota^- \rangle$

Relation r_ι^-: $\boxed{a_i = b_\iota}$

Relation r_ι^+: $\boxed{\tilde{a}_j = b_\iota}$

at each crossing ι:



$\tilde{a}_j = a_k \cdot a_j \cdot a_k^{-1}$ in $\pi_1(U)$

$\Rightarrow \quad \pi_1(U\cup V) = \langle a_i \mid r_\iota \rangle$ Relation r_ι $\boxed{a_k \cdot a_j \cdot a_k^{-1} = a_i}$

<u>Cor</u> $H_1(S^3\setminus K) \cong \mathbb{Z}^{|\pi_0(K)|}$

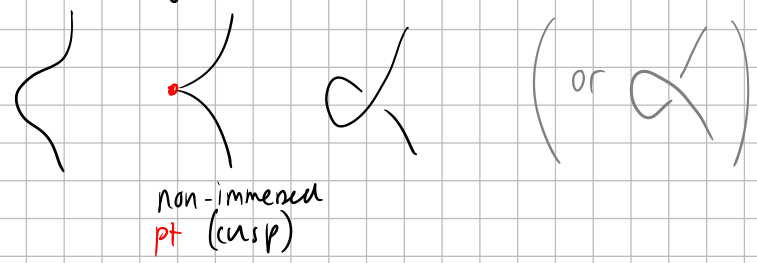
<u>Proof</u> <u>Exercise 39</u>

Reidemeister moves

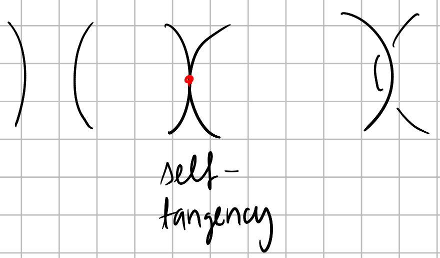
Thm (Reidemeister) For a generic perturbation of a smooth isotopy $\gamma_t : K \hookrightarrow \mathbb{R}^3$, $pr_{xy} \circ \tilde{\gamma}_t$ is an isotopy of knot diagrams, <u>except</u> for a finite number of times where one of the following transition occurs:

$y^2 = x^3 - 1 \qquad y^2 = x^3 \qquad y^2 = x(x^2-1)$

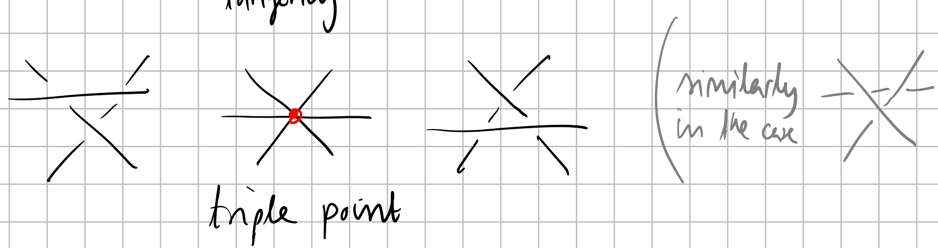
(R-I)



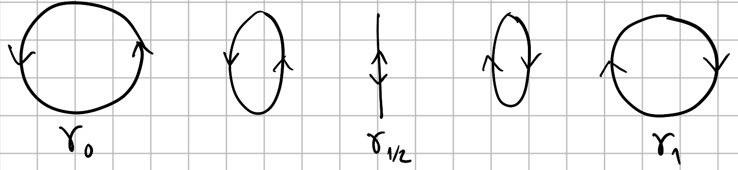
(R-II)



(R-III)

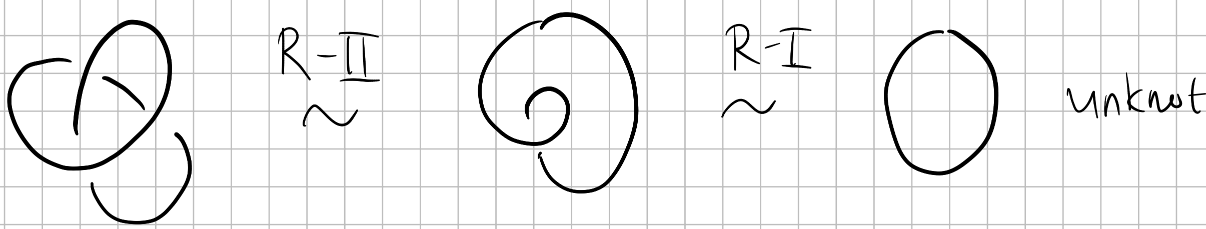


Rmk Turning around $S^1 \times \{0\}$ introduces bad singularities

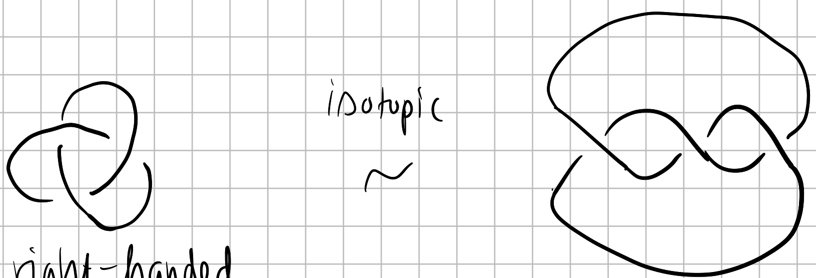


In fact, perturbing this family by a path of elements $A_t \in SO(3)$ does not help. (Lying in a linear plane is not generic)

Ex



Exercise 40 Show that



right-handed trefoil K_{tref}

Fact Coward & Lackenby gave an enormous upper bound on the neccessary Reidemeister moves needed to pass between to knot diagrams for the same knots (depending on the nr of crossings)

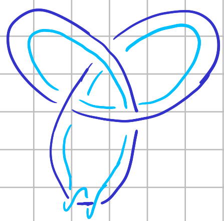
$\Rightarrow$ the question whether two knots are isotopic is decidable (at least in theory)

Ways to distinguish the trefoil & the unknot

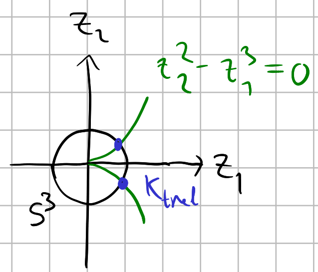
1.) (c.f. Lecture 11) Surgery on the <u>unknot</u> produces an S^1-bundle

$$S^1 \to E \to S^2 \quad \text{over } S^2$$

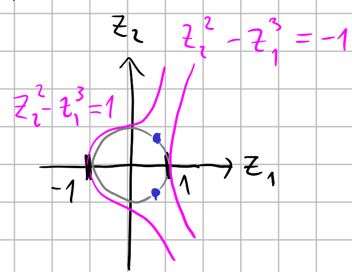
LES of π_i: $\underset{\mathbb{Z}}{\pi_1(S^1)} \twoheadrightarrow \pi_1(E) \to \underset{0}{\pi_1(S^2)} \Rightarrow \pi_1(E) = \mathbb{Z}/k\mathbb{Z}$ (abelian)

surgery on  produces the Poincaré homology sphere (whose π_1 is <u>not</u> abelian)

2.) $K_{tref} = \{(z_1, z_2) \in \mathbb{C}^2 \mid \underbrace{z_2^2 - z_1^3 = 0}_{\text{surface w. one singular pt}}\} \cap S^3$



$\Rightarrow S^3 \setminus K_{tref} \cong \{(z_1, z_2) \in \mathbb{C}^2 \mid \underbrace{z_2^2 - z_1^3 \in S^1}_{\text{fibration over } S^1 \text{ w. fibre } \mathbb{T}^2 \setminus \{pt\}}\}$



$$\Rightarrow S^3 \setminus K_{tref} \cong (\mathbb{T}^2 \setminus pt) \times [0,1] \Big/ (x,0) \sim (\varphi(x), 1)$$

"unwrapping" the above cylinder (c.f. $\mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow S^1$) yields a

$\varphi : \mathbb{T}^2 \setminus pt \xrightarrow{\cong} \mathbb{T}^2 \setminus pt$ cpt supp ($\varphi^6 = id$)

$\mathbb{Z}$ principal bundle

$$\mathbb{Z} \hookrightarrow (\mathbb{T}^2 \setminus \{pt\}) \times \mathbb{R} \twoheadrightarrow S^3 \setminus K_{tref}$$

LES of π_i

$\Rightarrow \pi_1((\mathbb{T}^2 \setminus \{pt\}) \times \mathbb{R}) \cong \langle a, b \rangle \hookrightarrow \pi_1(S^3 \setminus K_{tref})$

$\Rightarrow \pi_1(S^3 \setminus K_{tref}) \neq \mathbb{Z}$