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Knot invariants from the knot diagram

Colourability

A quandle is a pair $(Q, \triangleright)$ for which $\triangleright : Q \times Q \to Q$ satisfies

(Q1): $a \triangleright a = a$

(Q2): $\_ \triangleright a : Q \to Q$ bijective

(Q3): $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$ (right self-distributive)

(Q1) & (Q2): $a \triangleright b = b \iff a = b$ $(*)$

Ex 1.) $Q = G$ group $x \triangleright y := y^{-1} \cdot x \cdot y$

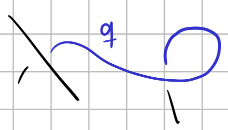
2.) $Q = \mathbb{Z}_p$, $p \geq 3$ prime

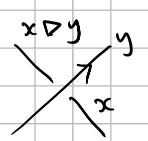
$\boxed{x \triangleright y := 2x - y}$ OBS: $x \triangleright y = x \iff y = x$ in this case

(Q3): $2(2a-b) - c = 4a - 2b - c$
$2(2a-c) - 2b + c = 4a - 2b - c$

$p=3$: $\triangleright$ characterised by $x \triangleright y = z$ either x, y, z all distinct or $x = y = z$

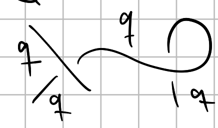
$\Rightarrow \triangleright$ commutative, $\mathbb{Z}_3 = \{R, G, B\}$ $R \triangleright R = R$ $R \triangleright G = B$

A Q-colouring of an oriented knot diagram is an assignment of $q \in Q$ to any arc 

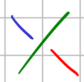
subject to the relation 

at each crossing.

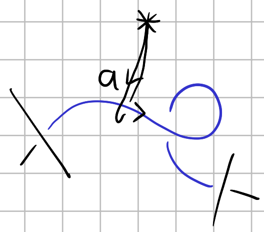
Obs The trivial colouring always exist for any $q \in Q$ & knot diagram. (Q1)



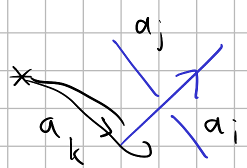
Ex $Q = \mathbb{Z}_3$: Colourings of the arcs in the knot diagram by RGB such that at each crossing either:

 all are same or  all different

Ex



A non-trivial $\pi_1(S^3 \setminus K)$-colouring for any K whenever $K \neq$ unknot (since π_1 not cyclic)



$$a_j = a_k^{-1} \cdot a_i \cdot a_k \quad \text{(see Lecture 15)}$$

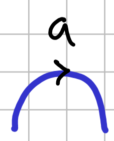
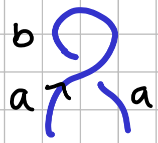
Thm The Reidemeister moves (R-I), (R-II), (R-III) induce natural bijections between Q-colourings that preserve the trivial colourings.

Cor The following are isotopy invariants for knots

- $\exists$ of non-trivial Q-colourings
- nr of Q-colourings

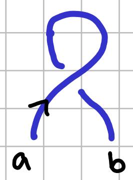
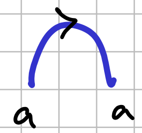
E.g. The unknot is the unique knot that only admits the trivial colourings for all Q.

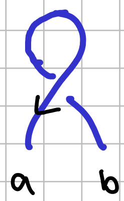
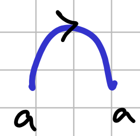
Proof

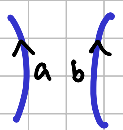
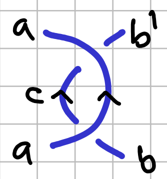
(R-I):  $\rightsquigarrow$ 

$a \triangleright a = b$
$\Rightarrow b = a$
(Q1)

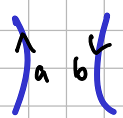
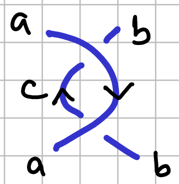
(analogously for opposite orientation)

 $\Rightarrow$ $b \triangleright a = a$
$(*) \Rightarrow b = a$ $\rightsquigarrow$ 

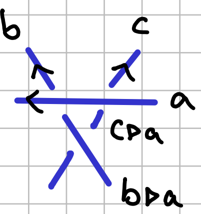
 $\Rightarrow b = a \triangleright a = a$ (Q1) $\rightsquigarrow$ 

(R-II):  $\leftrightsquigarrow$ 

$c = b \triangleright a$
$= b' \triangleright a$
(Q2) $\Rightarrow b = b'$

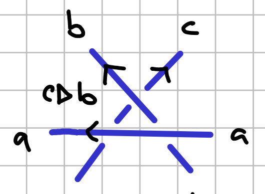
 $\leftrightsquigarrow$ 

c unique sol. to
$c \triangleright a = b$

(R-III): 

$(c \triangleright a) \triangleright (b \triangleright a)$

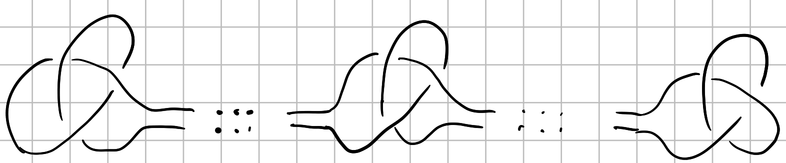
(etc...)



$(c \triangleright b) \triangleright a$
$\parallel$ (Q3)
$(c \triangleright a) \triangleright (b \triangleright a)$

$\square$

Exercise 41 Show that the k-fold connected sums

 K_k

of trefoils form an infinite family $\{K_k\}$ of pairwise non-isotopic knots, where no K_k is isotopic to the unknot. (Hint: use 3-colourings)

There are thus infinitely many embeddings $S^1 \hookrightarrow \mathbb{R}^3$ / isotopy

Alexander polynomial

We need
- linking numbers in S^3
- Seifert surfaces for knots

For an <u>oriented</u> link $K_1 \sqcup K_2 \subseteq S^3$ (K_i possibly consists of several components)

we can define the linking nr in several ways:

Gauß' definition (c.f. winding nr in Lecture 2)

Choose orientation preserving parametrisations γ_i of K_i.

$$\Gamma(\theta_1,\theta_2) := \gamma_1(\theta_1) - \gamma_2(\theta_2) : \underbrace{K_1 \times K_2}_{\text{union of tori}} \longrightarrow \mathbb{R}^3 \setminus \{0\}$$

$$lk(K_1,K_2) := \operatorname{wind}(\Gamma) = \oint_\Gamma F \cdot \bar{n}\, dS =$$

$$\boxed{= \frac{1}{4\pi} \int_0^{2\pi}\int_0^{2\pi} \frac{\Gamma(\theta_1,\theta_2)}{|\Gamma(\theta_1,\theta_2)|^3} \cdot \left(\dot\gamma_1(\theta_1) \times \dot\gamma_2(\theta_2)\right) d\theta_1\, d\theta_2}$$

Rmk $lk(K_1,K_2) \in \mathbb{Z}$ and is invariant under <u>homotopies</u> of $\Gamma : K_1 \times K_2 \to \mathbb{R}^3 \setminus \{0\}$ (or γ_i w. disjoint images).

One can also define linking using the knot diagram

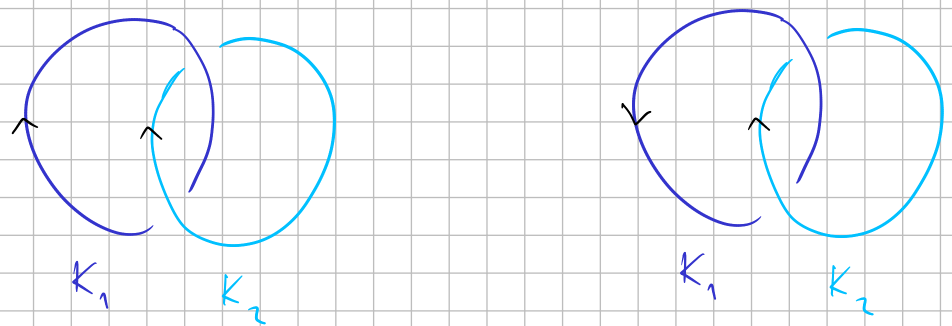
Thm

$$\boxed{lk(K_1, K_2) = \sum 1 + \sum (-1) \in \mathbb{Z}}$$

(Sum over all crossings with K_1 on top & K_2 below.)

Exercise 42 Show using the Reidemeister moves that the linking nr is invariant of smooth isotopy of $K_1 \sqcup K_2$.

Ex Hopf link (two fibres in the Hopf fibration $S^1 \hookrightarrow S^3 \twoheadrightarrow S^2$)



$lk(K_1, K_2) = +1$ $\qquad\qquad$ $lk(K_1, K_2) = -1$

Exercise 43 Show that $\boxed{lk(K_1, K_2) = lk(K_2, K_1)}$

and $\boxed{lk((-1)^i K_1, (-1)^j K_2) = (-1)^{i+j}\, lk(K_1, K_2)}$

change of orientation

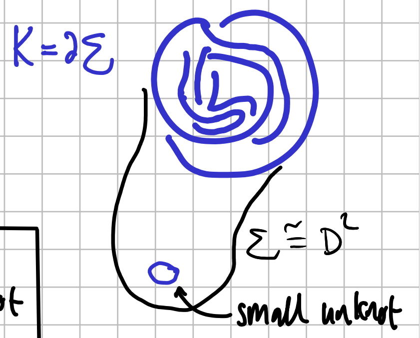
Hint: For first equality: rotate the link & use invariance.

Seifert surfaces

Def A Seifert surface for a knot $K \subseteq \mathbb{R}^3$ is a compact, connected, orientable surface $\Sigma \underset{\text{submfd}}{\subseteq} \mathbb{R}^3$ with boundary $\partial \Sigma = K$.

The Seifert genus $g(K)$ of K is the minimal genus of any of its Seifert surfaces.

(Any such $\Sigma \cong \Sigma_{g, k = |\pi_0(K)|}$)



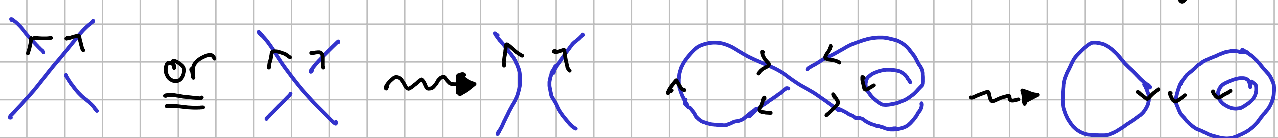
Clearly: $\boxed{|\pi_0(K)| = 1 : \quad g(K) = 0 \iff K \text{ unknot}}$

We will see that Seifert surfaces exist, so $g(L) \in \mathbb{Z}_{\geq 0}$,

How to construct a Seifert surface

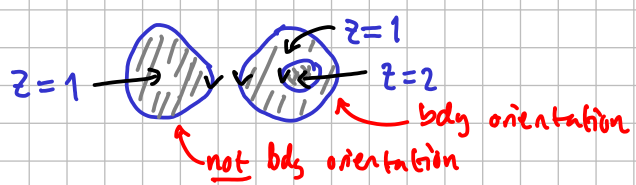
Take a knot diagram for a link $K \subseteq \mathbb{R}^3$. (The surface constructed will depend on this choice) | for knots: choice does not affect Σ.

Step (1): Choose an orientation & resolve crossings by

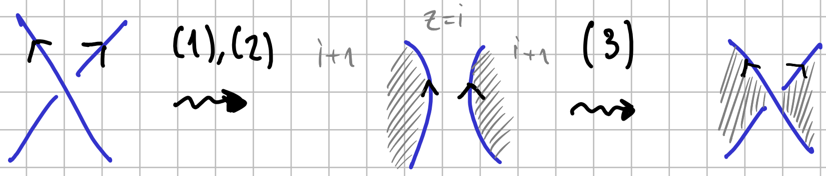


Step (2): We obtain $d > 0$ closed oriented curves that bound d nr. of nested discs $\subseteq \mathbb{R}^2$. (⚠ orientation might differ from bdy orientation)

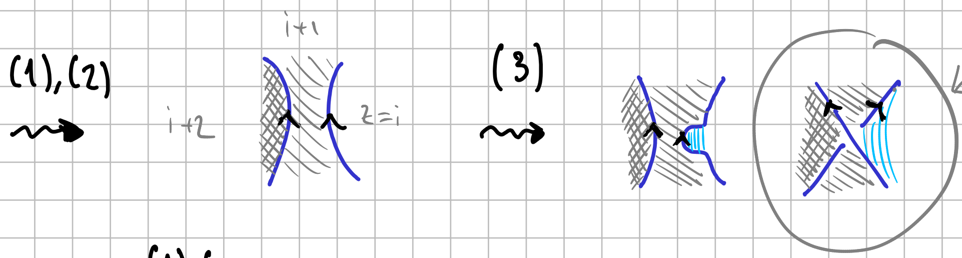
Lift the disc at the i:th level of the nesting to $\mathbb{R}^3$ by giving it coordinate $z = i$.



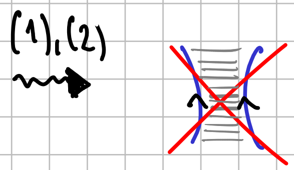
Step (3): Add a twisted band at each crossing according to



OR



This is a surface Σ with $\partial\Sigma = K$

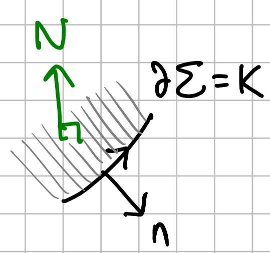


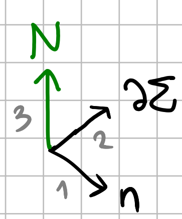
not possible since bdy orientation either agrees or disagrees w. any closed curve.

$\square$

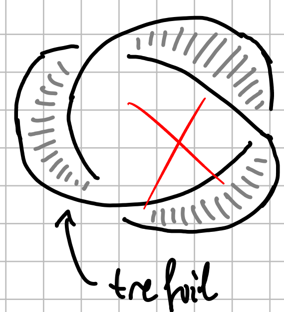
- The constructed surface Σ is connected when $\partial\Sigma = K$ is connected. Otherwise: connect by "tubes".

- It is orientable since the surface is two-sided, outward normal N can be assigned by



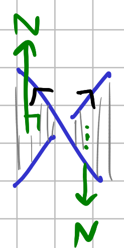


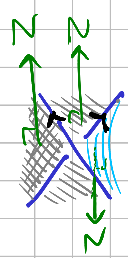
"right hand rule"



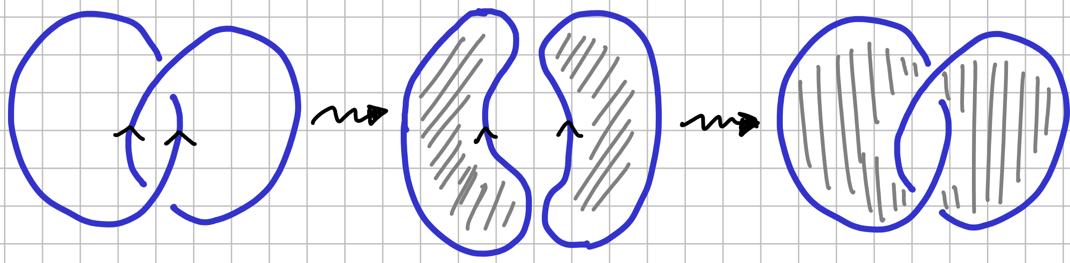
A Möbius band is one-sided and thus not a Seifert surface

It suffices to check that the orientation agrees near the bands



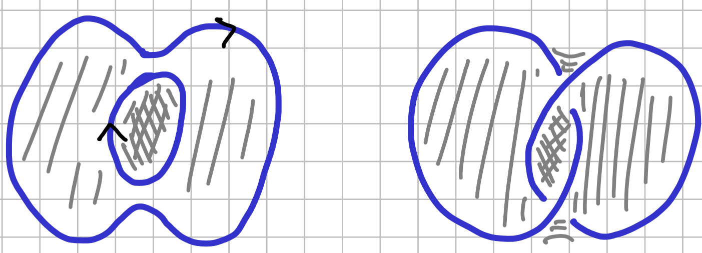


Ex Hopf link



(1) (2) (3)

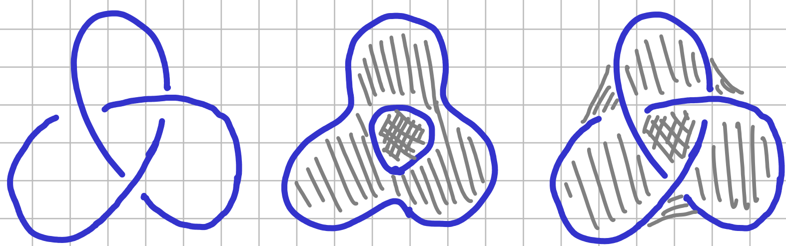
Σ = annulus



(1) (2) (3)

Σ = annulus

Ex Trefoil



(either orientation)

$g=1$ since
- oriented
- conn. bdy

Next time: how to find g.