L.17 PhD course on Geometry & Topology georgios.dimitroglou@math.uu.se

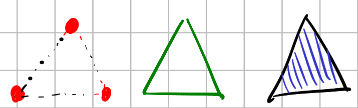
Knot polynomials

Before defining the Alexander polynomial.

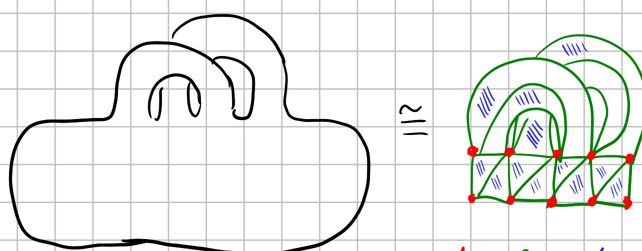
How to find the genus of a surface with boundary

Method 1 Triangulate Σ, consider the Euler characteristic

$$\boxed{\chi(\Sigma) = \#V - \#E + \#F = 2-2g-k}$$

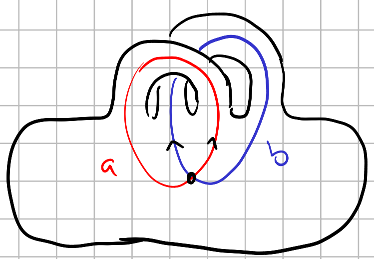


vertices edges faces



$10-23+12=-1$

Method 2

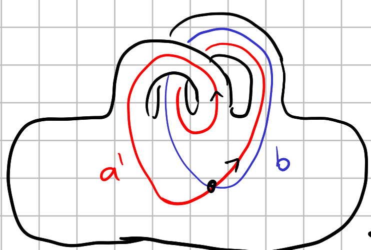


Σ_1

a, b form a basis of $H_1(\Sigma_{g,k}) = \mathbb{Z}^{2g} \oplus \mathbb{Z}^{k-1}$

g: genus

k: nr of boundary components



$\Sigma_{g,k}$

$a' = a+b$, b different basis

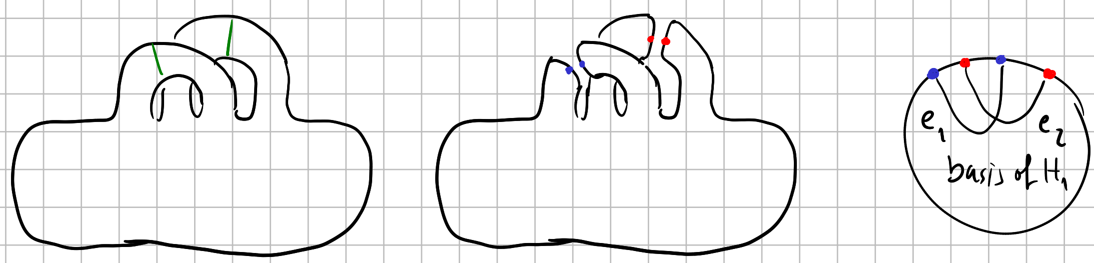
Our goal is to find the basis of simple closed curves as above for any Σ. But first, we need to find g.

(1) Cut $\Sigma_{g,k}$ open along disjoint embedded arcs with boundary $\subseteq \partial\Sigma$

Continue until we obtain a disc

(2) The number of arcs needed $= 2\cdot g + (k-1)$

Ex



$g=1$

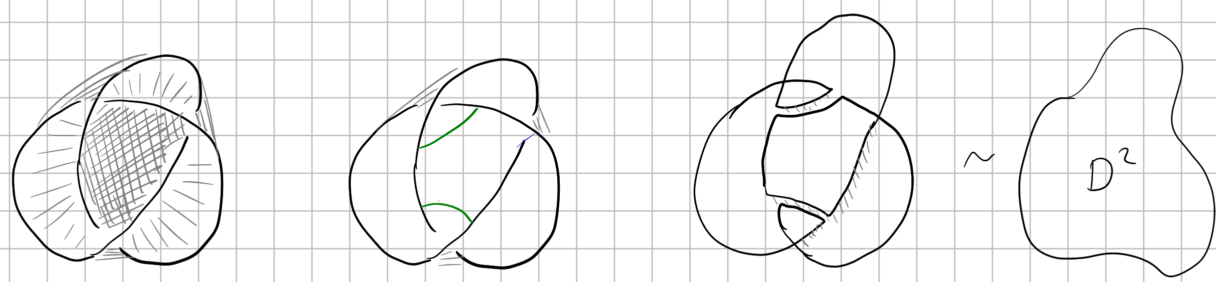
Rem • This reconstructs $\Sigma_{g,k}$ as one-handle attachments on D^2

See Lecture 10.

• We obtain a basis of H_1 by curves that intersect a unique arc transversely in a single point.

When $g>1$, the handle decomposition is unfortunately not unique (there are handle-slides).

Ex



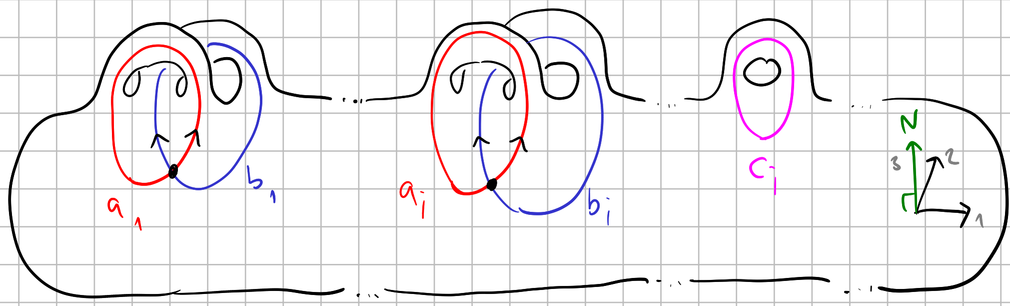
$\Sigma_{g,k}$ $k=1$: $2g+(k-1)=2 \Rightarrow g=1$

$\Sigma_{g,k}$ has a basis of $H_1(\Sigma_{g,k}) = \mathbb{Z}^{2g} \times \mathbb{Z}^{k-1}$ represented by simple closed curves

$$a_1, \dots, a_g, \; b_1, \dots, b_g, \; c_1, \dots, c_{k-1} \in H_1(\Sigma_{g,k})$$

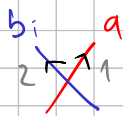
where moreover $x_i \cap x_j = \emptyset$ unless • $x_i = x_j$

or • $\{x_i, x_j\} = \{a_i, b_i\}$ single transverse intersection

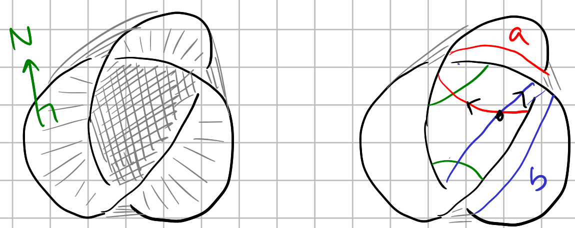


Given an orientation of Σ, we orient a_i & b_i so that the frame

$$\langle T_{a_i \cap b_i} a_i, T_{a_i \cap b_i} b_i \rangle = T_{a_i \cap b_i} \Sigma$$ agrees with the orientation of Σ.



Ex



⚠ For $g > 1$, the handle decomposition produced by the arcs need not give a basis of H_1 with the correct intersection properties.

Fact There is a geometrically defined bilinear intersection form

$$I : H_1(\Sigma) \otimes H_1(\Sigma) \to \mathbb{Z} \quad (\Sigma \rightsquigarrow \underset{\text{opposite orient.}}{-\Sigma} \text{ gives } I \rightsquigarrow -I)$$

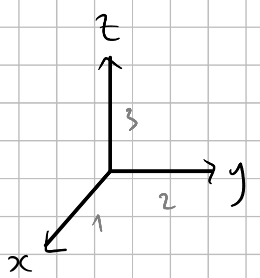
$$I(a_i, b_i) = 1 = -I(b_i, a_i)$$

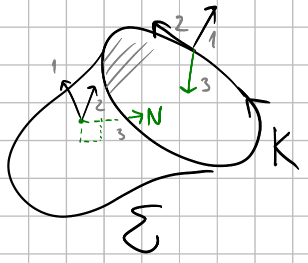
(orientability is crucial)

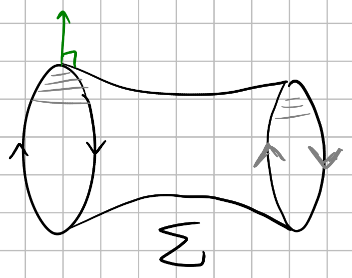
$\Rightarrow$ when $k=0$, $2g$ curves form a basis if they intersect like a's & b's

The Alexander polynomial

Let K be an <u>oriented</u> knot/link, and $\Sigma_{g,\pi_0(K)}$ a Seifert surfaces with orientation <u>induced by K</u>.







orientation determined by remaining bdy

- If Σ is a Seifert surface ($\Rightarrow$ conn. & orientable) then an orientation of a single component of K induces an orientation of the remaining components.
- The algorithm for constructing Σ from Lecture 16 depends on the orientation of K: it extends over the surface.

Choose a basis $e_1 = a_1, e_2 = b_1, e_3 = a_2, e_4 = b_2, \dots, e_{2g} = b_g$

$$e_{2g+1} = c_1, \dots, e_{2g+k-1} = c_{k-1}$$

of simple closed curves with the above intersection properties (depends on the orientation of Σ, and hence on that of K).

The Seifert matrix induced by Σ is

$$V \in \mathrm{Mat}_{2g+k-1,\,2g+k-1}(\mathbb{Z}) \text{ with entries}$$

$$\boxed{v_j^i = \mathrm{lk}(e_i, e_j^\#) \quad \Big| \quad \text{pushoff of } e_j \text{ along normal of } \Sigma.}$$

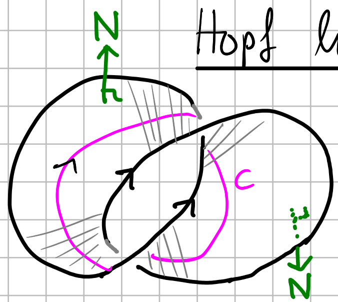
row

col.

V is invariant / change of basis & certain stabilisations, but we will not focus V itself here.

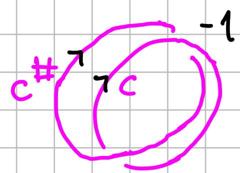
Ex

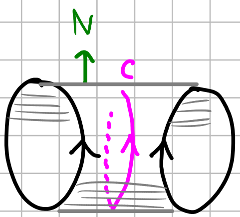
Hopf link



Σ = annulus (twisted)

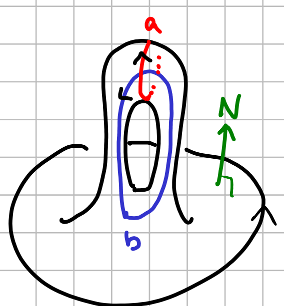
$$V = [\mathrm{lk}(c, c^\#)] = [-1]$$





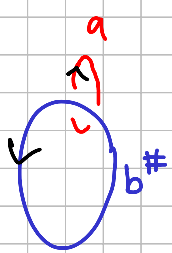
unlink $\quad V = [0]$

Σ = annulus (untwisted)



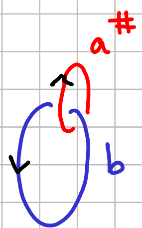
unknot

$\mathrm{lk}(a, b^\#)$



$$V = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$\mathrm{lk}(b, a^\#)$

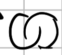


The Alexander polynomial is the Laurent polynomial

$$\boxed{\begin{array}{l} \Delta_K(t) := t^N \cdot \det\left(tV - V^{tr}\right) \in \mathbb{Z}[t,t^{-1}] \\[1em] N \in \mathbb{Z} \quad \text{s.t.} \quad \begin{cases} \bullet\ \Delta_K \ \underline{\text{polynomial}} & \text{(no negative } t \text{ powers)} \\ \bullet\ \Delta_K \text{ not divisible by } t & \text{(possibly zero)} \end{cases} \end{array}}$$

⚠
- there are different conventions for the normalisation
- $\det(0\times 0\text{-matrix}) = 1$

Ex

$\Delta_{\bigcirc\bigcirc}(t) = 0$, Δ  $(t) = 1 - t$,

$\Delta_{\bigcirc}(t) = t^{-1} \cdot \det\begin{bmatrix} 0 & -1 \\ t & 0 \end{bmatrix} = 1.$

Exercise 44 Show that (1.) $\pm t^{\deg \Delta} \Delta(t^{-1}) = \Delta(t)$

(2.) $\Delta_K(1) = \pm 1$ if $|\pi_0(K)| = 1$

Exercise 45 Calculate Δ  $(t) = 1 - t + t^2$

by using the above algorithm.

Thm (Alexander, '28) The Alexander polynomial $\Delta_K(t)$ is an invariant of oriented knots & links up to smooth isotopy.

Skein relations

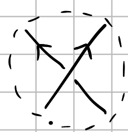
Some knot invariants can be determined via Skein-Relations.

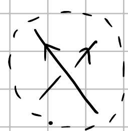
Conway found the following version of the Alexander polynomial in the '60s.

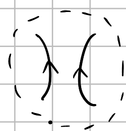
$$\nabla_K(z) \in \mathbb{Z}[z^{\pm 1}], \qquad \boxed{\Delta_K(t) = t^{M/2} \cdot \nabla_K(t^{1/2} - t^{-1/2})} \quad M \text{ suitable}$$

- ∇_K is also an invariant of oriented knots

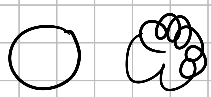
Conway showed $\boxed{\nabla_{K_+}(z) - \nabla_{K_-}(z) = z \cdot \nabla_{K_0}(z)}$


K_+

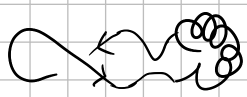

K_-

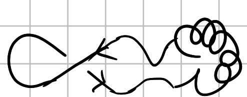

K_0

Which together with $\boxed{\nabla_0(z) = 1}$ determins the invariant uniquely

Prop If a knot K has a diagram  then $\nabla_K(t) = 0$.

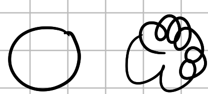
Proof

$K_+ =$ 

$K_- =$ 

isotopic $\Rightarrow \nabla_{K_+}(z) = \nabla_{K_-}(z)$

$\Rightarrow z \cdot \nabla_{K_0}(z) = 0$

$K_0 =$  $= K$ $\Rightarrow \nabla_{K_0}(z) = 0$ $\square$

How to compute $\nabla_K(t)$ using the Skein relation:

By induction, assume that $\Delta_K(t)$ has been computed for all knots that admit a knot diagram with $\leq n-1$ crossings.

Let K be a knot that admits a diagram with n crossings.

(1) For a suitable change $\times \rightsquigarrow \times$ at suitable subsets of the crossings, the new knot $\tilde{K}$ is isotopic to $\bigcirc\ \bigcirc \cdots \bigcirc$ (unlink).

<u>Proof</u> Make z-coordinate increasing everywhere except along some small outer <u>arc</u>. $\square$

(2) Change crossings of $\tilde{K}$ one at a time to get back K. Solve the unknown term in the skein relation inductively.