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## The Jones Polynomial ('84)

We start with Kauffman's version from '90.

The **Kauffman bracket** $\langle \text{diagram} \rangle \in \mathbb{Z}[A, A^{\pm 1}]$

(i) $\langle \bigcirc \rangle = 1$

(ii) $\langle \bigcirc \sqcup \underset{K}{\circledcirc} \rangle = (-A^2 - A^{-2}) \cdot \langle \underset{K}{\circledcirc} \rangle$

(iii) $\langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle )( \rangle$ (c.f. Skein relation)

The **Jones polynomial** (Kauffman's version) is

$$\boxed{X_K(A) := (-A)^{-3\underbrace{\left(\sum_{\times} 1 - \sum_{\times} 1\right)}_{\text{the "writhe"}}} \langle K \rangle \in \mathbb{Z}[A^{\pm 1}]}$$

which is an invariant of oriented links / isotopy.

(Unknown if it detects the unknot.)

It has a description as the "trace" of a representation of $Br_n$ built using the Temperley-Lieb algebra.

Jones' original version is $V_K(t) = X_K(t^{-1/4}) \in \mathbb{Z}[t^{\pm 1/2}]$

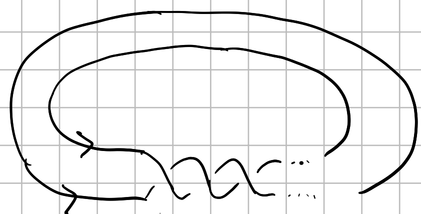
and is determined by the <u>Skein relation</u>

$$\boxed{t^{-1} V_{K_+}(t) - t \cdot V_{K_-}(t) = \left(t^{1/2} - t^{-1/2}\right) \cdot V_{K_0}(t), \quad V_O(t) = 1}$$

$K_+$ $K_-$ $K_0$

**Thm** $X_K$ & $V_K$ only depend on the smooth knot type isotopy class of the oriented knot $K$.

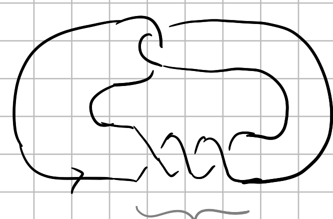
**Exercise 46.** Compute both the Alexander & Jones polynomials of the $(2,n)$-torus links



n-crossings (half-twists)

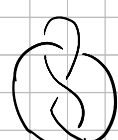
$n=1$: unknot, $n=2$: Hopf link
$n=3$: trefoil

**Exercise 47.** Compute both the Alexander polynomial & the Jones' polynomial by for the n-twist knots by using the Skein relation.



n crossings (half-twists)

$n=1$: trefoil $n=2$: "figure-8"



# § IV Connections & Gauge theory

In the 80's mathematics & physics showed that Gauge Theory,
i.e. the study of connections on a principal bundle,
give rise to many deep topological invariants.

SU(2)-principal bundles & connections $SU(2) \hookrightarrow E \twoheadrightarrow B$

$B = M^4$ Donaldson invariants of smooth structures

$B = M^3$ Instanton Floer homology, Chern-Simons thy

$B = S^3 \setminus K$ Jones polynomial (Witten)

flat connections $\longleftrightarrow$ path independent parallel transports
$dA + [A \wedge A] = 0$

$\Updownarrow$

group homom. $\rho : \pi_1(B) \to SU(2)$

Kronheimer-Mrowka '02 showed that
If $K \subseteq S^3$ not the unknot, then
$\exists \; \rho : \pi_1(S^3 \setminus K) \xrightarrow{\text{homom}} SU(2)$ with non-cyclic image

On a principal bundle we will investigate:

connections $\longleftrightarrow$ parallel transport

$\Updownarrow$

connection 1-form

flat connections $\longleftrightarrow$ parallel transp. indep. of path / homotopy

$\Updownarrow$

$dA + \frac{1}{2}[A \wedge A] = 0$

# Connections & Parallel transport

A choice of connection on a fibre bundle is equivalent to the choice of parallel transport.

The former is easier to formulate, but less easy to interpret.

**Def** A connection on a smooth fibre bundle $E \xrightarrow{p} B$ is a smooth choice of tangent subspaces $H_x \subseteq T_x E$, called horizontal subspaces, for each $x \in E$ subject to:

(A1) $D_x p \big|_{H_x} \longrightarrow T_{p(x)} B$ is a linear isomorphism ($\Rightarrow \dim H_x = \dim B$)

(A2) For any $v \in T_b B$, the lifts $\tilde{v}_x \in H$, $p(x) = b$, are of the form $\frac{d}{dt} \varphi_{\gamma(t)}(y_x)$

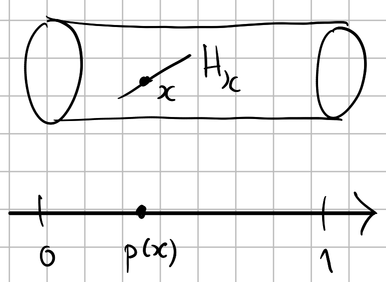
$$\varphi_{\gamma(t)} \in H^E_{\gamma(t)} = \{\varphi : F \to \gamma^{-1}(t)\}$$

preferred identifications see Lecture 4

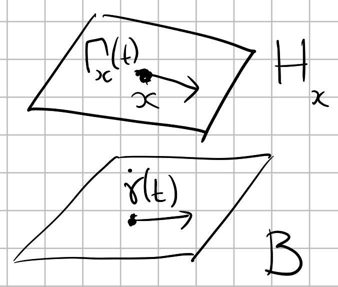
**Rem** When $E$ is a principal $G$-bundle, $H^E_b = \{\varphi_x(\cdot) = r_\cdot(x)\}$

(A2) becomes $\boxed{H_{x \cdot g} = D_x r_g (H_x)}$

smooth right $G$-action: $r : E \times G \longrightarrow E$
$(x, g) \longmapsto r_g(x) = x \cdot g$




$E = S^1 \times I$




For any smooth $\gamma : [0,1] \to B$, we can construct a uniquely defined smooth vector-field "over $\gamma$":

$\Gamma_x(t) \in T_x E$ defined for all $x$ s.t. $p(x) = \gamma(t)$, given by:

$$\boxed{\begin{array}{l} D_x p(\Gamma(t)) = \dot\gamma(t) \in T_{\gamma(t)} B \\ \Gamma(t) \in H_x \end{array}}$$

uses (A1) $\longrightarrow$ $\Gamma(t) \in H_x$

There is an induced ODE for any initial cond. $x \in p^{-1}(\gamma(0))$

$$\boxed{\begin{cases} \frac{d}{dt} \Pi_\gamma(x,t) = \Gamma_{\Pi_\gamma(x,t)}(t) \\ \Pi_\gamma(x,0) = x \end{cases}}$$

$\Rightarrow$ smooth isotopy $\boxed{\Pi_\gamma(t) : p^{-1}(\gamma(0)) \longrightarrow p^{-1}(\gamma(t))}$

called the <u>parallel transport</u>

Facts

- $\Pi_\gamma$ depends "smoothly" on $\gamma$
- $\Pi_\gamma(0) = id_{p^{-1}(\gamma(0))}$
- $\Pi_{cst}(t) \equiv id_{p^{-1}(cst)}$

- $\Pi_{\gamma\circ g}(t) = \Pi_\gamma(g(t))$, $g:[0,1]\xrightarrow{C^\infty}[0,1]$ } differentiate w.r.t. $t$
- $\Pi_{\gamma(1-\cdot)}(t)\circ\Pi_\gamma(1) = \Pi_\gamma(1-t) \Rightarrow$ } $\Pi_{\gamma(1-\cdot)}(1) = \Pi_\gamma(1)^{-1}$
- $\Pi_\gamma(1) = \Pi_{\gamma(\cdot - t_0)}(1-t_0)\circ\Pi_\gamma(t_0)$ (concatenation)

(A2)
$\Rightarrow$ • The preffered identification $H^E_b = \{\varphi : F \to p^{-1}(b)\}$
$\left(H^E_b \circ G = H^E_b\right.$; see Lecture 4$\left.\right)$ are preserved by $\Pi_\gamma$, i.e.

$$\boxed{H^E_{\gamma(1)} = \Pi_{\gamma(0)}(1)\circ H^E_{\gamma(0)}}$$

$\Rightarrow$ Parallel transport in well-defined also for piecewise smooth paths, e.g. concatenation

$$\gamma_0 * \gamma_1(t) = \begin{cases} \gamma_0(2t), & t\in[0,1/2] \\ \gamma_1(2t-1), & t\in[1/2,0] \end{cases}$$

For a $G$-principal bundle (A2) $\Rightarrow$ $\left(E\times G \to G \text{ right action transitive on fibres}\right)$
$\Pi_\gamma(t) : p^{-1}(\gamma(0)) \to p^{-1}(\gamma(1))$ <u>$G$-equivariant</u> diffeomorphism

When $\gamma(0) = \gamma(1)$ (loop)

After identifying

$$\begin{tikzcd} p^{-1}(\gamma(0)) \arrow[r, "\Pi_\gamma(1)"] \arrow[d, phantom, "\cong"] \arrow[d, phantom, "\varphi_x" left] & p^{-1}(\gamma(1)) \arrow[d, phantom, "\cong"] \arrow[d, phantom, "\varphi_x" left] \\ G \arrow[r, "\mu"'] & G \end{tikzcd}$$

$G$

$$\varphi_x(g) = x\cdot g \in p^{-1}(\gamma(0)), \quad x\in p^{-1}(\gamma(0))$$

$$\mu(g) = \mu(e\cdot g) = \mu(e)\cdot g \Rightarrow \mu = \ell_{\mu(e)}$$

When $G = O(n)$, $F = \mathbb{R}^n$

vector bundle w. orthogonal structure group.

$$\begin{array}{ccc} \Pi_\gamma : p^{-1}(\gamma(0)) & \longrightarrow & p^{-1}(\gamma(1)) \\ \wr\wr & & \wr\wr \\ \mathbb{R}^n & \xrightarrow[\in O(n)]{} & \mathbb{R}^n \end{array}$$

$\longleftarrow$ admissible choice

for any choice of admissible trivialisation.

**Thm** The following properties are equivalent:

1) $\Pi_{\gamma_0}(1) = \Pi_{\gamma_1}(1)$ when $\gamma_0 \sim \gamma_1$ are homotopic rel. endpoints
2) $\Pi_\gamma(1)$, $\gamma(0) = \gamma(1)$, only depends on $[\gamma] \in \pi_1(B, \gamma(0))$
3) $\Pi_\gamma(1) = id_{p^{-1}(0)}$ whenever $\gamma(0) = \gamma(1)$ & $\gamma$ has image in a chart $\cong \mathbb{R}^n$
4) $\{\Pi_\gamma\}_{\gamma(0)=\gamma(1)} \subseteq \mathrm{Diff}^\infty(p^{-1}(\gamma(0)))$ is a countable subgroup

In this case, we say that the connection is flat

((1) $\Leftrightarrow$ repr. of the fundamental groupoid)