 georgios.dimitroglou@math.uu.se

Thm The following properties are equivalent:

1) $\Pi_{\gamma_0}(1) = \Pi_{\gamma_1}(1)$ when $\gamma_0 \sim \gamma_1$ are homotopic rel. endpoints
2) $\Pi_\gamma(1)$, $\gamma(0)=\gamma(1)$, only depends on $[\gamma] \in \pi_1(B, \gamma(0))$
3) $\Pi_\gamma(1) = \mathrm{id}_{p^{-1}(0)}$ whenever $\gamma(0)=\gamma(1)$ & γ has image in a chart $\cong \mathbb{R}^n$
4) $\{\Pi_\gamma(1)\}_{\gamma(0)=\gamma(1)} \subseteq \mathrm{Diff}^\infty(p^{-1}(\gamma(0)))$ is a countable subgroup

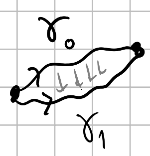
In this case, we say that the connection is flat

((1) $\Leftrightarrow$ $\gamma \mapsto \Pi_\gamma(1)$ is a representation of the fundamental groupoid of B)

Proof 1) $\Rightarrow$ 2) $\Rightarrow$ 3) obvious

↑ charts are contractible, $\Pi_{cst}(1) = \mathrm{id}_{p^{-1}(cst)}$

2) $\Rightarrow$ 1)



$\gamma_0 * (\gamma_1(1-\cdot)) = 0$ in $\pi_1(B, \gamma(0))$

$\Leftrightarrow$ $\gamma_0 \sim \gamma_1$ rel. endpoints

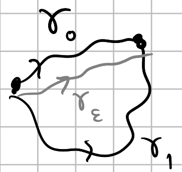
$\gamma_0 * (\gamma_1(1-\cdot))$

$\gamma_1 * (\gamma_1(1-\cdot)) \sim cst_{\gamma(0)}$

On one hand: $\Pi_{\gamma_0 * (\gamma_1(1-\cdot))}(1) = \Pi_{cst_{\gamma_0}}(1) = \mathrm{id}_{cst_{\gamma_0}}$ by 2)

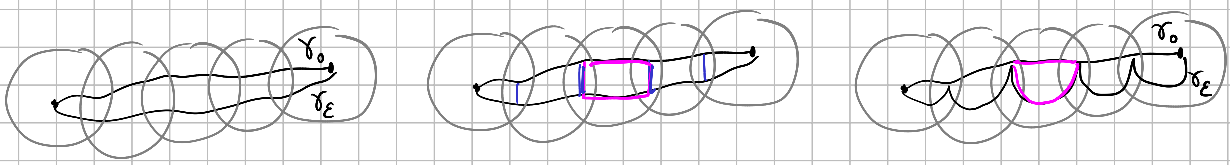
on the other $\Pi_{\gamma_0 * (\gamma_1(1-\cdot))}(1) = \left(\Pi_{\gamma_1}(1)\right)^{-1} \circ \Pi_{\gamma_0}(1)$

3) $\Rightarrow$ 1). By above, 1) holds for paths that live in a <u>single local chart</u>

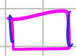
Assume two general paths  are homotopic rel. endpoints.

w.l.o.g: $\gamma_s(t_0)$ & $\gamma_{s+\varepsilon}(t_0)$ live in the same chart for all fixed $t_0 \in [0,1]$

(divide the homotopy in small t-steps) Cover the paths by finitely many charts

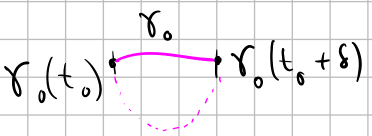
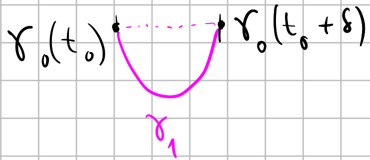


Cover γ_0 & γ_ε by $\mathbb{R}^n$-charts

 loop contained inside a chart

deform γ_ε rel. endpoints. use 1) in the loc. charts

Again use 1) in local charts to deduce that parallel transport

along  or  are equal.

4) $\Leftrightarrow$ 2) roughly holds by continuity of $\gamma \rightsquigarrow \Pi_\gamma(1)$

2) $\Rightarrow$ 4) $\pi_1(B,*)$ countable since it is the path components of a second countable space

4) $\Rightarrow$ 2) any map into a countable subset of a metric space must be constant on path components.

$\square$

Flat Bundles

By the above result, any fibre bundle $F \hookrightarrow E \overset{p}{\twoheadrightarrow} B$ w. flat connection $H \subseteq TE$ gives rise to a canonical group morphism ρ^H

$$\pi_1(B,*) \ni [\gamma] \mapsto \Pi_\gamma(1) \in \mathrm{Diff}^\infty(p^{-1}(\gamma(0)))$$

$\Vert$S using $H^E_{\gamma(0)} = \{\varphi : F \overset{\cong}{\to} p^{-1}(\gamma_0)\} = \varphi \circ G$

$$\rho^H : \pi_1(B,*) \to G < \mathrm{Diff}^\infty(F)$$

ρ^H well-defined up to a conj. by $g \in G$ ($p^{-1}(\gamma(0)) \cong F$ non canonical ⚠)

Obs When $E \twoheadrightarrow B$ is a flat G-principal bundle,

$$2) \Rightarrow \Pi_\cdot(1) : \pi_1(B,*) \to \mathrm{Diff}^\infty(p^{-1}(*))$$

takes values in $\varphi_x \circ G \circ \varphi_x^{-1}$ for some choice $x \in p^{-1}(*)$, where $\varphi_x(g) = x \cdot g$, $\varphi_x \in H^E_*$ and the subgroup $G \hookrightarrow \mathrm{Diff}^\infty(G)$ is induced by $g \mapsto \ell_g$

Obs $(\varphi_{x\cdot g} = \varphi_x \circ \ell_g)$

$$(\varphi_{x\cdot g})^{-1} \circ \varphi \circ \varphi_{x\cdot g} = \ell_{g^{-1}} \circ \underbrace{(\varphi_x^{-1} \circ \varphi \circ \varphi_x)}_{\ell_h} \circ \ell_g, \quad \varphi \in \mathrm{Diff}^\infty(p^{-1}(*))$$

In conclusion: For a flat principal G-bundle, there is a induced map $\rho = \Pi_\cdot(1) : \pi_1(B,*) \longrightarrow G$ which is uniquely determined up to conjugation.

Thm Two flat bundles (E_i, H_i) are isomorphic iff

$$\rho^{H_0} = g \cdot \rho^{H_1} \cdot g^{-1}$$

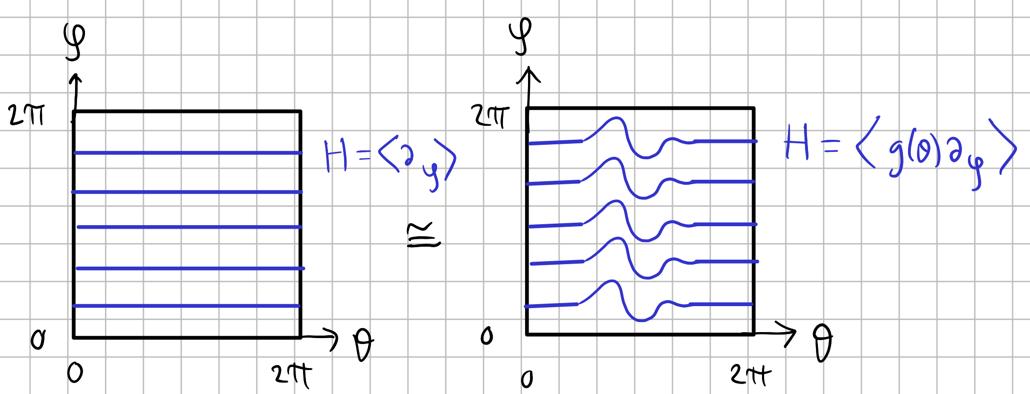
differ by conjugation with some $g \in G$

Proof **Exercise 48** Hint: Use parallel transport to define the isom. □

Cor If $\pi_1(B) = 0$, then all flat bundles are trivial.

(E.g. there are no flat connections on $TS^2 \twoheadrightarrow S^2$
or $S^1 \hookrightarrow S^3 \twoheadrightarrow S^2$)

Ex Non-equal but isomorphic connections on $S^1_\varphi \times S^1_\theta \overset{p}{\twoheadrightarrow} S^1_\theta$

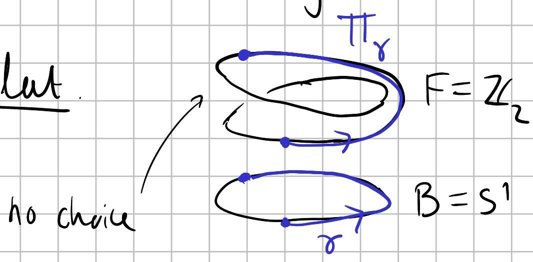


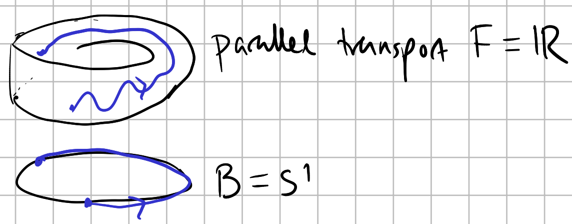
$\Pi_\gamma(1) = id$

Next lecture we will introduce the group of Gauge transformations. One is mainly interested in connections / equivalence under Gauge transformation.

When F is an abelian group, e.g. $F = \overbrace{\mathbb{Z}_m, \mathbb{Z}}^{G \text{ discrete}}, \overbrace{\mathbb{R}^n, \mathbb{C}^n}^{G = Gl_n}$ then we call E equipped with a flat connection a <u>local system on B</u>

<u>Exercise 49</u> Show that E has a unique parallel transport when F is discrete. By condition (3) above, this is obviously <u>flat</u>.



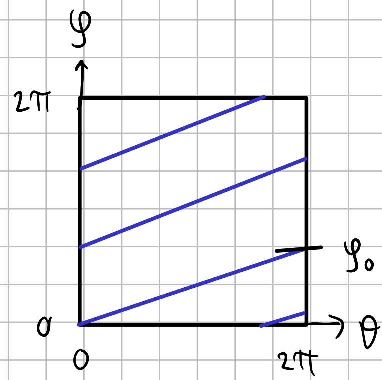


<u>Ex</u> • Any $e^{i\varphi_0} \in U(1)$ gives rise to a local system with fibre $\mathbb{C}$ on S^1:

$$E = \Big(\underset{(z\ ,\ \theta)}{\mathbb{C} \times [0, 2\pi]} \Big) \Big/ (z, 0) \sim (\underset{U}{e^{i\varphi_0}} \cdot z, 2\pi) \quad \big(\Rightarrow \Pi_{S^1}(1) = e^{i\varphi_0} \big)$$

• similarly $\quad S^1_\varphi \times [0, 2\pi] / \sim \quad S^1$-bundle w. connection

Since $U(1)$ is abelian, different φ_0 give non-isomorphic conn.



<u>Facts</u>

- All bundles $E \twoheadrightarrow S^1$ w. $G = U(1)$ are trivial <u>as bundles</u> (see Lecture 7)
- All connections on S^1 are flat since $\dim B = 1$
- Hence, any connection is det. by the phase/monodromy $\varphi \in [0, 2\pi)$

Recall LES of fundamental groups

$$\pi_i(F,*) \to \pi_i(E,*) \xrightarrow{p_*} \pi_i(B,p(*)) \xrightarrow{\delta_i} \pi_{i-1}(F,*)$$

A covering space of B is the same as a connected fibre bundle $E \xrightarrow{p} B$ with discrete fibre F.

Ex $(2\pi\mathbb{Z})^n \hookrightarrow \mathbb{R}^n \twoheadrightarrow \mathbb{T}^n = (S^1)^n$ ($\mathbb{R}^n$ connected)

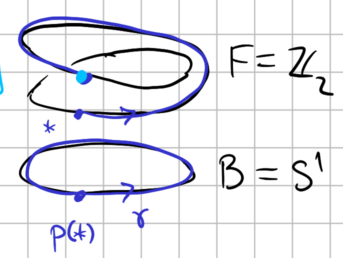
$x \longmapsto x \bmod 2\pi$

$\pi_j(F) = 0 \quad j > 0 \Rightarrow p_*$ iso. when $i > 1$
inj of groups when $i = 1$,

For $i = 1$:

$$\pi_1(E,*) \hookrightarrow \pi_1(B,p(*)) \overset{\delta_1}{\twoheadrightarrow} \pi_0(F,*) \to \pi_0(E,*) = 0 \text{ (by assumption)}$$

Obs $\boxed{\delta_1[\gamma] = \Pi_\gamma(1)\,(*) \in \pi_0(F,*) \cong F}$



$E \cong S^1 \Rightarrow$ covering space

Cor The quotient δ_1 (not a group morphism in general), or equivalently the subgroup $\pi_1(E,*) \hookrightarrow \pi_1(B,p(*))$, classifies the covering space E up to bundle isomorphism.

Proof δ_1 determines π_1 since $\Pi_{\gamma_1}(1)(\delta_1(\gamma_0)) = \Pi_{\gamma_0 * \gamma_1}(1)$
& since δ_1 is surjective onto $\pi_0(F,*) \cong F$. □

The Lie bracket

In order to use analytic methods for obtaining flat connections, we need to introduce forms.

Lie algebras

$\mathrm{Diff}^\infty(M)$ is a "Frechét Lie group"

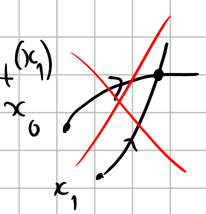
$T_{id}\left(\mathrm{Diff}^\infty(M)\right) = \Gamma(TM)$ smooth vector fields on TM

(they generate smooth 1-param. families of diffeomorphisms, i.e smooth isotopies)

$X \in \Gamma(TM)$ smooth vector-field

$$\text{ODE} \begin{cases} \varphi_0^X(x) = x \\ \frac{d}{dt}\varphi_t^X(x) = X_{\varphi_t^X(x)} \end{cases} \longleftrightarrow \text{ smooth 1-parameter families } \varphi_t^X \in \mathrm{Diff}^\infty(M) \text{ (isotopies)}$$

$\varphi_t(x_0) = \varphi_t(x_1)$



Uniqueness & smoothness to solutions of ODE's

$\Rightarrow$

- φ_t^X diffeomorphism (when defined)
- $\varphi_{t_0}^X \circ \varphi_{t_1}^X = \varphi_{t_0+t_1}^X$ (differentiate time to get "=")

i.e. φ_t^X 1-dim subgroup of $\mathrm{Diff}^\infty(M)$ (abelian $\cong (\mathbb{R}, +)$)

Conjugation: $\varkappa_{(\varphi_t^X)^{-1}}(\varphi_s^Y) = (\varphi_t^X)^{-1} \circ \varphi_s^Y \circ \varphi_t^X = \varphi_{-t}^X \circ \varphi_s^Y \circ \varphi_t^X$

$\Rightarrow$ t parametrises a smooth family of subgroups; $t=0$: φ_s^Y

when φ^X_t & φ^X_s commute, $t \mapsto \varkappa_{\varphi^X_{-t}}(\varphi^Y_s) \equiv \varphi^Y_s$ is constant

In general, the vector field corr. to $s \mapsto \varphi^X_{-t} \circ \varphi^Y_s \circ \varphi^X_t$

is given by $Y^t = (\varphi^X_{-t})_* Y \overset{\text{def}}{=} \frac{d}{dt}\left(\varphi^X_{-t} \circ \varphi^Y_s \circ \varphi^X_t\right)$ $\quad (Y^0 = X)$

The infinitesimal non-commutativity is measured by

$$\frac{d}{dt}\frac{d}{ds}\left(\varphi^X_{-t} \circ \varphi^Y_s \circ \varphi^X_t\right)\Big|_{s=t=0} = \frac{d}{dt}(\varphi^X_{-t})_* Y\Big|_{t=0} =: [X, Y] \in \Gamma(TM)$$

$\nwarrow$ Lie bracket

this is the Lie derivative of Y in the direction of X

Exercise 50 Show that, in local coordinates $x_1, \dots, x_n$,

where $X_{\bar{x}} = \sum_{i=1}^n X^i(\bar{x})\partial_{x_i}$, $Y_{\bar{y}} = \sum_{i=1}^n Y^i(\bar{x})\partial_{x_i}$, we have

- $$[X,Y]_{\bar{x}} = \sum_{i=1}^n \left(D(Y^i)(X_{\bar{x}}) - D(X^i)(Y_{\bar{x}})\right)\partial_{x_i}$$
- $$df[X,Y] = d(df(Y))(X) - d(df(X))(Y)$$

  for any C^∞ map $f: \mathbb{R}^n \to \mathbb{R}$
- $[X,Y] = -[Y,X]$ (anti-commutativity)
- $[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$ (Jacoby identity)

Hence, the vector field $\Gamma(TM)$ on M form an ∞-dimensional Lie-algebra ($[-,-]$ is obviously $\mathbb{R}$-linear)