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Lie algebras of Lie groups

G Lie group, $g \in G$, $\quad l_g(x) = g \cdot x$, $\quad r_g(x) = x \cdot g$, $\quad r_g, l_g \in \mathrm{Diff}^\infty(G)$

$\mathrm{Diff}^\infty(G)$ is an ∞-dim Lie group with multiplication $= \circ$

$$\begin{aligned} G &\longrightarrow \left(\mathrm{Diff}^\infty(G), \circ^{op}\right) \\ g &\longmapsto r_g \end{aligned} \qquad \text{(see Lecture 5)}$$

A diffeomorphism $\varphi \in \mathrm{Diff}^\infty(G)$ is <u>left-equivariant</u> if

$$l_h \circ \varphi = \varphi \circ l_h \quad \text{for all } h \in G$$

Since l_h acts transitively: $\varphi = r_g$ for some $g \in G$ $\Rightarrow$

<u>left-equivariant one parameter subgroups</u> $\varphi_t^X < \mathrm{Diff}^\infty(M)$ generated by $X \in \Gamma(TG)$ are of the form $r_{g^X(t)}\, G \to G$ for some 1-parameter subgroup $(\mathbb{R},+) \longrightarrow (G,\cdot)$ of G.

$$t \longmapsto g^X(t)$$

$$l_h \circ \varphi_t^X = \varphi_t^X \circ l_h \iff Tl_h(X) = \frac{d}{dt}\left(l_h \circ \varphi_t^X\right)\Big|_{t=0} = \frac{d}{dt}\left(\varphi_t^X \circ l_h\right)\Big|_{t=0} = X_{l_h}$$

$$\iff (l_h)_*(X) = T_{(l_h)^{-1}}\, l_h(X) = X$$

Hence, the left-equivariant subgroups correspond to the <u>left-invariant</u> vector-fields

$$\mathfrak{g} := \left\{ X \in \Gamma(TG) \;\middle|\; (l_h)_* X = X \text{ for all } h \right\} \cong T_e G$$

Recall from last time that $\Gamma(TM)$ is an ∞-dim Lie algebra

$$\boxed{[X,Y] \overset{\text{def}}{=} \frac{d}{dt}\frac{d}{ds} \varkappa_{\varphi^X_{-t}}(\varphi^Y_s)\Big|_{\substack{t=0\\s=0}} = \frac{d}{dt}(\varphi^X_{-t})_*(Y)\Big|_{t=0}}$$

$[\_,\_] : \Gamma(TM) \otimes_{\mathbb{R}} \Gamma(TM) \longrightarrow \Gamma(TM)$ $\mathbb{R}$-bilinear, antisymmetric & satisfies Jacobi identity

Prop (1) For any diffeomorphism $f \in C^\infty(M,N)$

$$f_*[X,Y] = [f_*X, f_*Y]$$

(2) If G is a Lie group, then $\mathfrak{g} \subseteq \Gamma(TG)$ is a Lie-subalgebra.

Proof (1.): $f_*\left(\frac{d}{dt}(\varphi^X_{(-t)})_* Y\right) \overset{\text{Chain rule}}{=} \frac{d}{dt}\left((f\circ\varphi^X_{(-t)})_* Y\right)\Big|_{t=0} =$

$$= \frac{d}{dt}\left((f\circ\varphi^X_{(-t)}\circ f^{-1})_* f_* Y\right)\Big|_{t=0} \overset{(\dagger)}{=} [f_*X, f_*Y]$$

$(\dagger)$ $f_*Z = T_{f^{-1}}f(Z) = \frac{d}{dt}(f\circ\varphi^Z_t\circ f^{-1})\Big|_{t=0} \Rightarrow \varphi^{f_*Z}_t = f\circ\varphi^Z_t\circ f^{-1}$

(2): If $X,Y \in \mathfrak{g} \Rightarrow (\ell_h)_*[X,Y] \overset{(1.)}{=} [(\ell_h)_*X, (\ell_h)_*Y] = [X,Y]$

$\square$

The adjoint representation is the left G-representation

$$\mathrm{Ad}_g \overset{\text{def.}}{=} (\varkappa_g)_* = T_{\varkappa_{g^{-1}}}\varkappa_g(\_)\big|_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathfrak{g}$$

$$\varkappa_g(x) = \{x \mapsto g \cdot x \cdot g^{-1}\} = l_g \circ r_{g^{-1}} = r_{g^{-1}} \circ l_g$$

$$(l_h)_* \circ (\varkappa_g)_*(X) = (r_{g^{-1}})_* \circ (l_h)_* \circ (l_g)_* X = (r_{g^{-1}})_* X = (r_{g^{-1}})_* \circ (l_g)_* = (\varkappa_g)_*(X)$$

$X \in \mathfrak{g}$

Prop

$$Ad_g = (r_{g^{-1}})_* \big|_{\mathfrak{g}}$$

i.e. $(Ad_g X)_{x \cdot g^{-1}} = Tr_{g^{-1}}(X_x)$

Hence • Ad_g preserves $\mathfrak{g}$

- $Ad_g = (r_{g^{-1}})_* \big|_{\mathfrak{g}} \iff (Ad_g X)_{x \cdot g^{-1}} = Tr_{g^{-1}}(X_x)$
- $Ad_g [X, Y] = [Ad_g X, Ad_g Y]$
- Since $\varkappa_g \circ \varkappa_h = \varkappa_{g \cdot h}$, the chain rule gives

  $Ad_g \circ Ad_h = Ad_{g \cdot h}$ $\leftarrow$ acts linearly on $\mathfrak{g}$

  hence it is a left G-representation in $Gl(\mathfrak{g})$.

In particular: $\boxed{Ad_g [X, Y] = [Ad_g(X), Ad_g(Y)]}$

$$\boxed{ad_X(Y) \overset{\text{def}}{=} \frac{d}{dt} Ad_{g^X(t)}(Y) = \frac{d}{dt} \left(\varkappa_{g^X(t)}\right)_*(Y) = [X, Y]}$$

To conclude, we have produced the adjoint representations

$$\boxed{\begin{array}{ll} \mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g} & [\cdot,\cdot]\text{-preserving left } G\text{-representation} \\ \mathrm{ad}_X Y = [X,Y] & \mathfrak{g}\text{- Lie algebra representation} \\ \text{i.e. } \mathrm{ad}_{[X,Y]}(z) = \mathrm{ad}_X \circ \mathrm{ad}_Y(z) - \mathrm{ad}_Y \circ \mathrm{ad}_X(z) & \end{array}}$$

by Jacobi $[[X,Y],z] = [X, \mathrm{ad}_Y z] - [Y, \mathrm{ad}_X z]$

Rem G abelian: $\mathrm{Ad}_g \equiv \mathrm{id}_{\mathfrak{g}}$, $[-,-] \equiv 0$, $\mathrm{ad} \equiv 0$

Exercise 51 When $G = Gl_n$ (over $\mathbb{R}$ or $\mathbb{C}$), show that

$$\boxed{\begin{array}{l} \mathfrak{g} \cong \mathrm{Mat}_{n,n} \\ \mathrm{Ad}_g(X) = g \cdot X \cdot g^{-1}, \quad X \in \mathrm{Mat}_{n,n} \\ \mathrm{ad}_X(Y) = X \cdot Y - Y \cdot X \end{array}}$$

where "$\cdot$" is ordinary matrix multiplication

Connection one form

Def A connection one-form on a principal G-bundle $G \hookrightarrow E \twoheadrightarrow B$ is a $\mathfrak{g}$-valued 1-form $A \in \Gamma(\mathrm{Hom}(TE,\mathfrak{g})) = \Gamma(T^*E \otimes_{\mathbb{R}} \mathfrak{g})$ for which

(B1) $A \circ T\varphi_x(X_e) = X \in \mathfrak{g}$ $\quad \varphi_x : G \xrightarrow{\cong} p^{-1}(p(x)) \subseteq E$, $\; g \mapsto x \cdot g$ $\quad$ canonical parametrisation

(B2) $R_g^* A \overset{\text{def}}{=} A \circ DR_g = \mathrm{Ad}_{g^{-1}} \circ A$, $\quad R_g(x) = x \cdot g$ $\quad$ right G-action, $x \in E$, $g \in G$

G-equivariance

Said differently, TE-valued 1-form $\in \Gamma(\mathrm{Hom}(TE, TE))$ which is

- vertical (values in $\ker(TE \to TB)$)

(B2) • G-equivariant, and

(B1) • satisfies $\quad A \circ T\varphi_x|_{\mathfrak{g}} = \mathrm{id}_{\mathfrak{g}} \qquad \varphi_x : \begin{array}{l} G \hookrightarrow E \\ g \mapsto x \cdot g \end{array}$ canonical param

Rmk • (B1) is compatible with (B2)

$$A \circ T(R_g \circ \varphi_x) = A \circ T(\varphi_x \circ r_g) \underset{(\mathrm{Prop})}{=} A \circ T\varphi_x \circ \mathrm{Ad}_{g^{-1}} \underset{(B1)}{=} \mathrm{Ad}_{g^{-1}} \circ A$$

• $X \in \mathfrak{g}$:

$$A \circ T\varphi_x(X_h) = A \circ T\varphi_x(T\ell_h X_e) = A \circ T\varphi_{x \cdot h}(X_e)$$

i.e. $A \circ T\varphi_x(X_h)$ = the unique extension of $X_h \in T_h G$ a vector-field in $\mathfrak{g}$ (i.e. left-inv.)

Ex For $F = E = G$, $B = pt$, this is the Cartan one-form $\quad \theta \in \Gamma(T^*G \otimes_{\mathbb{R}} \mathfrak{g})$

Obs: $\theta(X_g) = \theta(X_e) = X$ when $X \in \mathfrak{g}$

Exercise 52 Show that there is a bijection between connections $H \subseteq TE$ and connection one-forms A^H on a principal G-bundle, determined by

$$\boxed{A\left(\frac{d}{dt}\Pi_\gamma(t)(x)\right) = 0} \quad \text{for any } x \in p^{-1}(\gamma(0))$$

Exercise 53 Show that the space of connections is non-empty, and forms an affine space (any choice of origin A_0 makes it a vector space).

Curvature two-form

The curvature two form is given by

$$\boxed{F_A \overset{\text{def}}{=} dA + A \wedge A \in \Omega^2(M) \otimes \mathfrak{g}}$$

This is a $\mathfrak{g}$-valued (antisymmetric) two-form with values in $\mathfrak{g}$

The wedge product of $\mathfrak{g}$-valued forms is given by (obs: not anti-comm!)

$$(A \wedge B)(V_1, V_2) = \frac{1}{2}\Big(\big[A(V_1), B(V_2)\big] - \big[B(V_1), A(V_2)\big]\Big) \overset{A=B}{=} \big[A(V_1), A(V_2)\big]$$

F_A G-equivariant since • $R_g^* dA = d R_g^* A = d\, Ad_{g^{-1}} A = Ad_{g^{-1}} A$

• $[R_g^* A(V_1), R_g^* A(V_2)] = [Ad_{g^{-1}} A(V_1), Ad_{g^{-1}} A(V_2)] = Ad_{g^{-1}} [A(V_1), A(V_2)]$

$$dA(V_1, V_2) = \big(D A(V_2)\big) V_1 - \big(D A(V_1)\big) V_2 - A[V_1, V_2] \quad \text{Cartan's formula}$$

Ex When $E = G$, $\boxed{F_\theta = d\theta + \theta \wedge \theta = 0}$ by Cartan's formula.

$$\big(V_i \in \mathfrak{g} \Rightarrow A(V_i) \equiv V_i\big)$$

Gauge transformations $\Psi \in \mathcal{G}(E)$: G-equivariant $\Psi \in \mathrm{Diff}^\infty(E)$ (Gauge group)

$$\begin{tikzcd} E \arrow[r, "\Psi", "\cong"'] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[d, "\cong"] \\ B \arrow[r, "id_B"'] & B \end{tikzcd}$$

In any local trivialisation $U \times G$ D

$$\Psi(u, g) = (u, h(u) \cdot g), \quad h: U \to G \text{ smooth}$$

Equivalently: $\Psi(x) = R_{\psi(x)} \qquad \psi : E \longrightarrow G$

$$\Psi(x\cdot g) = \Psi(x)\cdot g \Rightarrow \psi \circ R_g = \kappa_{g^{-1}} \circ \psi$$

Obs: Locally, Gauge transformations are of the form

$$\mathcal{U}\times G \longrightarrow \mathcal{U}\times G$$
$$(u,g) \longmapsto (u, h(u)\cdot g) \qquad \Rightarrow \qquad \psi(u,g) = g^{-1}\cdot h(u)\cdot g$$

Prop $\Psi^* A = (R_{\psi(x)})^* A = Ad_{\psi^{-1}} \circ A + \psi^* \vartheta$

In local coordinates we thus get:

$$(g^{-1}\cdot h\cdot g)^* \vartheta = (l_{g^{-1}} \circ h \circ r_g)^* \theta = r_g^* \circ h^* \circ l_{g^{-1}}^* \vartheta$$

$$= \underbrace{Ad_{g^{-1}} \circ \vartheta(\underbrace{dh}_{\in T_h G})}_{\text{extension to v.f. in } \mathfrak{g}}$$

Thm $F_{\Psi^* A} = Ad_{\psi^{-1}} \circ F_A$

Proof $d\Psi^* A + \Psi^* A \wedge \Psi^* A$

$$= d\left(Ad_{\psi^{-1}} \circ A + \psi^* \vartheta\right) + \left(Ad_{\psi^{-1}} \circ A + \psi^* \vartheta\right) \wedge \left(Ad_{\psi^{-1}} \circ A + \psi^* \vartheta\right)$$

$$= Ad_{\psi^{-1}} F_A + \underbrace{\cancel{\psi^*(\vartheta + \vartheta \wedge \vartheta)}}_{=0 \text{ by Cartan}} + \underbrace{d(Ad_{\psi^{-1}}) \circ A + (Ad_{\psi^{-1}} \circ A) \wedge \psi^* \vartheta}_{=0 \text{ shown below}}$$

$$d\left(\mathrm{Ad}_{\psi^{-1}}\right)\circ A + \left(\mathrm{Ad}_{\psi^{-1}}\circ A\right)\wedge \psi^*\vartheta = \mathrm{ad}_{-\psi^*\vartheta}\left(\mathrm{Ad}_{\psi^{-1}}\circ A\right) + \left(\mathrm{Ad}_{\psi^{-1}}\circ A\right)\wedge\psi^*\vartheta = 0 \qquad \square$$

Thm The space of flat connections on $E \twoheadrightarrow B$ / Gauge transformation is equal to the finite-dimensional space of group morphisms

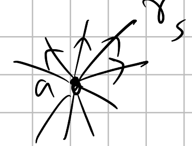
$$\rho : \pi_1(B) \longrightarrow G$$

since π_1 is finitely generated

up to conjugation. (See Lecture 19)

Thm $F_A = 0 \iff A$ is a flat connection

Proof ($\Leftarrow$): $\gamma_s(t)$ $\gamma_s(0) \equiv a$



Use $\Pi_{\gamma_s}(t)$ to construct a local trivialisation $B^n \times G$ that is parallel along radial lines $pt \in S^{n-1}$, $[0,1)\cdot pt \subseteq B^n$.

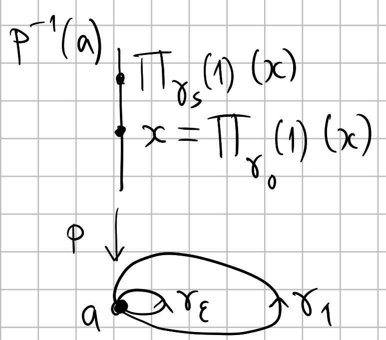
Flatness $\Rightarrow$ $\Pi_{\tilde\gamma}(1) = \mathrm{id}_G$ in these coordinates for any path $\tilde\gamma$ in B^n

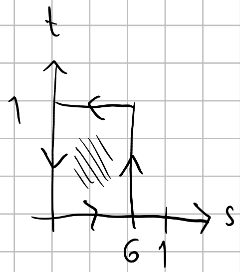
$$\Rightarrow H = \ker\left(pr_{B^n}\right) = TB^n \times 0 \subseteq T(B^n \times G)$$

$$\Rightarrow A = \vartheta \Rightarrow F_A = d\vartheta + \vartheta\wedge\vartheta = 0 \quad \text{(Cartan)}$$

($\Rightarrow$): Consider a smooth homotopy $\Gamma(s,t) : [0,1]\times[0,1] \to B$ of paths where $\gamma_s(t) = \Gamma(s,t)$, and $\gamma_0(t) \equiv a \equiv \gamma_s(1) \equiv \gamma_s(0)$

Consider $(s,t) \mapsto P(s,t) := \Pi_{\gamma_s}(t)(x)$, $x \in p^{-1}(a)$, i.e. the parallel transport of x along the t-directions





$$0 = \int_{[0,\sigma]\times[0,1]} p^*(F_A) \overset{\text{(Stokes)}}{=} \int_{[0,1]\times[0,1]} d\,p^*(A) = \cancel{\int_{P(s,0)} A} - \int_{P(s,1)} A + \cancel{\int_{P(\sigma,t)} A} - \int_{P(0,t)} A$$

$A\left(\frac{\partial P}{\partial t}\right) = 0$

$= 0$ $(P(s,0) \equiv x)$ $\qquad$ $= 0$ (parallel in t-direction)

$P(s,1) \in p^{-1}(a)$ for all $s \in [0,1]$

$$0 = \int_{P(s,1)} A = \int_0^\sigma A\left(\frac{d}{ds} P(s,1)\right) ds \overset{\frac{d}{d\sigma}}{\Rightarrow} A\left(\frac{d}{ds} P(s,1)\right) \equiv 0$$

$$\Rightarrow P(s,1) \equiv x$$

(Thm Lecture 19) $\Rightarrow$ flatness $\square$

Exercise 53 Show that when $G = U(1)$, the integral $\int_\gamma A \in \mathbb{R} = \mathfrak{g}$ computes the phase shift for the parallel transport along a loop $\gamma : [0,1] \to B$ $\gamma(0) = \gamma(1)$.