Heegaard Floer Homology

Motivation & overview

- Y connected, oriented closed mfd
- $-Y$ same mfd w. orientation reversed
- $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$
- H_* H^* are w. integer coeff. unless specified

$Y \rightsquigarrow \widehat{HF}(Y)$, $HF^+(Y)$, $HF^-(Y)$, $HF^\infty(Y)$

Lagrangian Floer homology

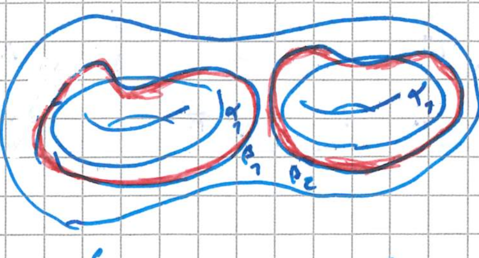
$Y \rightsquigarrow$ pointed Heegaard diagram $(\Sigma, \underline{\alpha}, \underline{\beta}, z)$

Σ closed, connected, and oriented surface of genus $g > 0$

$\underline{\alpha} = (\alpha_1, \dots, \alpha_g)$, $\underline{\beta} = (\beta_1, \dots, \beta_g)$ g-tuple of pairwise disjoint & embedded curves, each cut Σ into a planar surface.

z basepoint, $z \in \Sigma \setminus (\alpha_1 \cup \dots \cup \alpha_g \cup \beta_1 \cup \dots \cup \beta_g)$

S^3



$(S^2 \times S^1) \# (S^2 \times S^1)$

$(\Sigma, \underline{\alpha}, \underline{\beta}, z) \rightsquigarrow$

- symplectic manifold $\mathrm{Sym}^g \Sigma = \Sigma^g / S_g$ $\quad S_g$ perm. gp.
- symplectic codim-2 submanifold (divisor) $V_z = \{z\} \times \mathrm{Sym}^{g-1}(\Sigma)$
- Lagrangian tori $\mathbb{T}_\alpha = \alpha_1 \times \dots \times \alpha_g$

$$\mathbb{T}_\beta = \beta_1 \times \dots \times \beta_g$$

$\widehat{CF}(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ as Lagrangian Floer homology of $\mathbb{T}_\alpha, \mathbb{T}_\beta$ inside $\mathrm{Sym}^g \setminus V_z$

$CF^\circ(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ $\circ \in \{+, -, \infty\}$ inside $\mathrm{Sym}^g \Sigma$

differ by how curves intersected w. V_z are counted

$\widehat{CF}(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ is a finitely dim. ch. complex $/\mathbb{F}$

$CF^\circ(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ are complexes over $\mathbb{F}[U]$

<u>Tautological exact seq</u>:

$$0 \to CF^- \to CF^\infty \to CF^+ \to 0$$

$$0 \to \widehat{CF} \to CF^+ \xrightarrow{U} CF^+ \to 0$$

$$0 \to CF^- \xrightarrow{U} CF^- \to \widehat{CF} \to 0$$

Thm (Ozsváth - Szabó):

Heegaard diagrams of the same Y yield q.i. chain complexes.

Thm (Juhász - D. Thurston - Zemke): the map induced in homology is canonical

$\widehat{HF}(Y)$, $HF^{\pm}(Y)$, $HF^{\infty}(Y)$ for the homology of $\widehat{CF}(\Sigma, \underline{\alpha}, \underline{\beta}, z)$

Structural properties

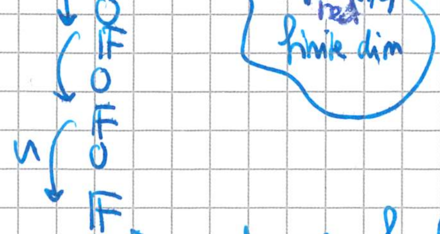
$$\widehat{HF}(Y) = \bigoplus_{s \in Spin^c(Y)} \widehat{HF}(Y, s)$$ and same for $+, -, \infty$

$Spin^c(Y) \cong H^2(Y)$

$\widehat{HF}(Y,s)$, $HF^{+}(Y,s) \neq 0$ only for finitely many $Spin^c$-structures.

If $H^1(Y, \mathbb{Q}) = 0$ ($\mathbb{Q}HS$):

- groups are $\mathbb{Q}$-graded
- U has degree -2
- in each $Spin^c$ structure
- $\chi(\widehat{HF}(Y,s)) = 1$ in each s



degree of bottom is $d(Y)$ d-invariant so $\dim_{\mathbb{F}} \widehat{HF}(Y, s) \geq 1$ for every $s \in Spin^c(Y)$ ($\mathbb{Q}$HS)

<u>Def</u> Y is an L-space if
- Y is a $\mathbb{Q}$HS
- $\dim_{\mathbb{F}} \widehat{HF}(Y, s) = 1$ for all $s \in Spin^c(Y)$

in particular: $L(p,q)$ are L-spaces

If $\pi_1(Y)$ is finite, then Y is an L-space.

<u>Conjecture</u> (Boyer-Gordon-Watson):

The same are equivalent for $\mathbb{Q}$HS Y:

- Y is not an L-space
- Y admits a taut foliation $\Uparrow$ OZ
- $\pi_1(Y)$ is left orderable
- verified (by the work of many people) for graph manifolds.

TQFT-like properties

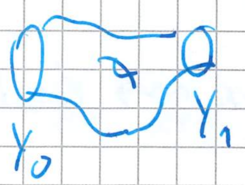
- $\widehat{HF}_*(-Y) \cong \widehat{HF}^*(Y) = \text{dual of } \widehat{HF}_*(Y)$
- $HF^{\pm}_{*}(-Y) \cong HF^{*}_{\mp}(Y)$

Connected sums

- $\widehat{HF}(Y_1 \# Y_2) \cong \widehat{HF}(Y_1) \otimes \widehat{HF}(Y_2)$
- $HF^-(Y_1 \# Y_2) \simeq CF^-(Y_1) \otimes_{\mathbb{F}[U]} CF^-(Y_2)$

  for suitable Heegaard diagrams

- W is a <u>connected</u> oriented cobordism from Y_0 to Y_1



$\partial W = Y_1 \sqcup -Y_0$

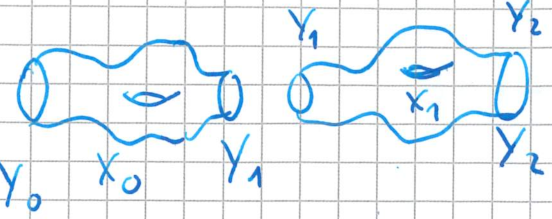
Then there are maps

$$\widehat{F}_W : \widehat{HF}(Y_0) \to \widehat{HF}(Y_1)$$

$$HF^{\circ}(Y_0) \to HF^{\circ}(Y_1) \quad \text{for } \circ = +, -, \infty$$

split according to Spin^c structures.

Composition formula:



$$X = X_0 \underset{Y_1}{\cup} X_1$$

$\hat{F}_X = \hat{F}_{X_1} \circ \hat{F}_{X_0}$, same for $+, -, \infty$

$K \subseteq Y$ knot $\rightsquigarrow (\Sigma, \underline{\alpha}, \underline{\beta}, w, z)$

Use $(\Sigma, \alpha, \beta, w)$ to build $CF(\Sigma, \alpha, \beta, w)$,

extra divi V_z to weigh the holomorphic discs

If K is null-homologous, this produces a filtration

in $CF(\Sigma, \underline{\alpha}, \underline{\beta}, w)$

take graded complex $\rightsquigarrow$ Knot Floer homology groups

$\widehat{HFK}(Y, K, d)$, similarly for other flavours

$\widehat{HFK}(Y, K) = \bigoplus_d$ ___ $HF(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ in $\mathrm{Sym}^g \Sigma \setminus (V_z \cup V_w)$

If $Y = S^3$, $\rightsquigarrow \widehat{HFK}(K, d) \leftarrow$ alexander grading

$$\subseteq F_{d-1} \subseteq F_d \subseteq F_{d+1}$$

different from 0 at finitely many gradings.

Basic properties

- symmetry $\widehat{HFK}(K,d) \cong \widehat{HFK}(K,-d)$

- $\sum_{d\in\mathbb{Z}} \chi(\widehat{HFK}(K,d))\, t^d = \Delta_K(t)$

Thm (O-Z) $g(K) = \max \{ d \mid \widehat{HFK}(K,d) \neq 0 \}$

Corollaries: $\widehat{HFK}$ detects the unknot, if $p^{\in\mathbb{Z}}$-surgery on K gives p-surgery on unknot, then K is the unknot.

Thm (G. for $g=1$, Ni for $g>1$) If K has genus g then K is fibred iff $\widehat{HFK}(K,-g) \cong \mathbb{F}$

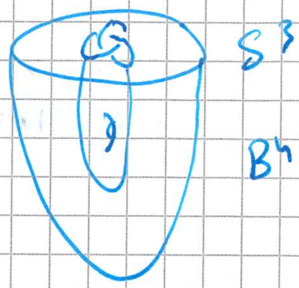
Cor
- $\widehat{HFK}$ detects the trefoil & figure 8
- if surg. on K gives the Poincaré homology sphere then K = 
- if surgery on K gives $L(p,q)$ then K is fibred

$K \rightsquigarrow$ filtration $\ldots \overset{\{0\}}{CF_{d-1}} \subseteq CF_d \subseteq \overset{\widehat{CF}(S^3)}{F_{d+1}} \subseteq \ldots$ in $\widehat{CF}(S^3)$

$(id)_* : H_*(F_d) \to \widehat{HF}(S^3) = \mathbb{F}$ (O-Sz)

$\tau(K) = \min\{ d \mid (id)_* \text{ is surjective}\}$

Thm $|\tau(K)| \leq g_4(K)$



ξ is a **contact structure** on Y ($\xi = \ker \alpha$, $\alpha \wedge d\alpha = vol$)

defines $c(\xi) \in \widehat{HF}(-Y)$

Thm (O-Sz)

If ξ is overtwisted (i.e $\exists D \hookrightarrow Y$ s.t. $\xi|_{\partial D} = TD|_{\partial D}$)

If ξ is (strongly) fillable then $c(\xi) \neq 0$

Lisca - Stipsicz: $\exists$ nonfillable ξ with $c(\xi) \neq 0$.

Classified Seifert mfds which do not admit tight cont. str

Q: $\exists \xi$ tight but w. $c(\xi) = 0$

Contact invariant is functorial under Weinstein cob.

Thm (G): $\exists\ \xi$ which is symplectically fillable but not Weinstein fillable.