L - II

O-Sz 1, Section 2.2

McDonald, Symmetric products of an algebraic curve

Σ Riemann surface

d-fold symmetric product $\mathrm{Sym}^d\Sigma$

$\mathrm{Sym}^d\Sigma = \Sigma^d / S_d \leftarrow$ permutation gp of d letters.

$\mathrm{Sym}^d\Sigma$ top. space quotient topology

$\underline{\Sigma = \mathbb{C}}$

<u>Prop</u> $\mathrm{Sym}^d\Sigma \cong \mathbb{C}^d$

$$\begin{tikzcd}
\left\{\begin{matrix}\text{monic polynomials} \\ \text{of deg } d\end{matrix}\right\} \arrow[r, "\text{coeff}"] \arrow[d, "\text{roots}"] & \mathbb{C}^d \arrow[dl, leftrightarrow] \\
\mathrm{Sym}^d\Sigma &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{C}^d \arrow[r, "\tilde{S}"] \arrow[d, "\pi"'] & \mathbb{C}^d \\
\mathrm{Sym}^d\mathbb{C} \arrow[ur, "S"'] &
\end{tikzcd}$$

$\tilde{S}(\underline{x}) = (s_1(x), \dots, s_d(x))$

elem. symm. funct

def. by $\prod_{i=1}^{d} (t + x_i) = \sum_{i=0}^{d} s_i(x)\, t^{d-i}$, $s_0 \overset{\text{def.}}{\equiv} 1$

S <u>cont. & invertible</u>

$$s_j(x_1, \dots, x_d) = \sum_{1 \le i_1 < \dots < i_j \le d} x_{i_1} \cdots x_{i_j}$$

Claim If the coeff. are bounded, the roots are bounded

$\Rightarrow$ S^{-1} continuous, because the inverse of a cont. invertible function on a compact subset is continuous.

Proof of claim $f(t) = \sum a_i t^{d-i}$ $\quad a_0 = 1$, $|a_i| < r$

$$|f(t)| \geq |t|^d - \sum |a_i| |t|^{d-i} \geq |t|^d - r \sum_{i=1}^{d} |t|^{d-i} \xrightarrow[t \to +\infty]{} +\infty$$

$\Delta \subseteq \mathrm{Sym}^d \mathbb{C}$ "big diagonal" i.e. where two components are equal

$\tilde{\Delta} \subseteq \mathbb{C}^d$ "big diagonal" $\pi^{-1}(\Delta) = \tilde{\Delta}$

$\mathbb{C}^d \setminus \tilde{\Delta} \longrightarrow \mathrm{Sym}^d \mathbb{C} \setminus \Delta$ covering

we push forward the complex structure on $\mathbb{C}^d \setminus \tilde{\Delta}$ to $\mathrm{Sym}^d \mathbb{C}$

$S: \mathrm{Sym}^d \mathbb{C} \longrightarrow \mathbb{C}^d$ holomorphic on $\mathrm{Sym}^d \mathbb{C} \setminus \Delta$

Lemma S is a local biholomorphism on $\mathrm{Sym}^d \mathbb{C} \setminus \Delta$

Proof It is enough to show that $\tilde{S}$ is a loc. bihol. on $\mathbb{C}^d \setminus \tilde{\Delta}$.

$\mathbf{x} = (x_1, \dots, x_d)$, $\quad \hat{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d)$

$$\tilde{S}(x) = (s_1(x), \ldots, s_d(x))$$

$d=3$

$$\tilde{S}(x) = (x_1 + x_2 + x_3,\ x_1 x_2 + x_2 x_3 + x_1 x_3,\ x_1 x_2 x_3)$$

$$d_x \tilde{S} = \left(s_{i-j}(\hat{x}_j) \right)_{\substack{1 \leq i \leq d \\ 1 \leq j \leq d}}$$

$$d_x \tilde{S} = \begin{pmatrix} 1 & x_2 + x_3 & x_2 x_3 \\ 1 & x_1 + x_3 & x_1 x_3 \\ 1 & x_2 + x_3 & x_2 x_3 \end{pmatrix}$$

$\det(d_x \tilde{S})$ polynomial of deg $\frac{d(d-1)}{2}$

if $x_i = x_j$ two rows become equal:

$$\det(d_x \tilde{S})\Big|_{x_i = x_j} = 0$$

$$\Rightarrow \underbrace{\prod_{i<j} x_i - x_j}_{\text{deg. } \frac{d(d-1)}{2}} \text{ divides } \det(d_x \tilde{S})$$

$$\Rightarrow \det(d_x \tilde{S}) = \alpha \prod_{i<j} x_i - x_j$$

$$\boxed{\alpha \neq 0}$$

$\Omega \subseteq \mathbb{C}$ open set, $\varphi : \Omega \to \mathbb{C}$ holomorphic

$$\begin{tikzcd} \mathrm{Sym}^d \varphi : \mathrm{Sym}^d \Omega \arrow[r] \arrow[d, "s"] & \mathrm{Sym}^d \mathbb{C} \arrow[d, "s"] \\ \Omega' \arrow[r] & \mathbb{C}^d \end{tikzcd}$$

$\mathrm{Sym}^d(\varphi)[(x_1, \ldots, x_d)] = [(\varphi(x_1), \ldots, \varphi(x_d))]$

$$\Omega' = s(\mathrm{Sym}^d(\Omega))$$

Lemma Φ is holomorphic

We know that Φ is holomorphic on $\Omega' \setminus \underbrace{S(\Delta)}$

Φ is cont. on Ω'

zero set of the discriminant (polynomial fcn)

Riemann extension theorem $\Rightarrow$ Φ is holomorphic everywhere

$S(\Delta) \subseteq \mathbb{C}^d$ zero set of polynomial.

Φ loc. bounded

Example If φ is C^∞ but not hol., Φ need not be smooth.

$\varphi : \mathbb{C} \to \mathbb{C}$, $\varphi(x+iy) = (2x + 1/2\, y)$

$\mathrm{Sym}^2(\varphi) : \mathrm{Sym}^2 \mathbb{C} \to \mathrm{Sym}^2 \mathbb{C} \rightsquigarrow \Phi$ not smooth

$$
\begin{tikzcd}
\mathbb{C}^2 \arrow[r, "\varphi \times \varphi"] \arrow[d] & \mathbb{C}^2 \arrow[d] \\
\mathrm{Sym}^2 \mathbb{C} \arrow[r, "\mathrm{Sym}^2 \varphi"] \arrow[d, "S"'] & \mathrm{Sym}^2 \mathbb{C} \arrow[d, "S"] \\
\mathbb{C}^2 \arrow[r, "\Phi"] & \mathbb{C} \\
\mathbb{R} \arrow[u] \arrow[ur, "\phi \circ \gamma"'] \arrow[uu, bend left, "\tilde{\gamma}"]
\end{tikzcd}
$$

$S(z_1, z_2) = (z_1 + z_2, z_1 z_2)$

$\gamma(t) = (0, t)$

$\tilde{\gamma}(t) = [(\sqrt{t}, -\sqrt{t})]$

If $t>0$ $\varphi(\sqrt{t}) = 2\sqrt{t}$

$t<0$ $\varphi(\sqrt{t}) = \frac{1}{2}\sqrt{t}$

So $$\underset{\underset{\displaystyle\Phi\circ\gamma(t)}{\|}}{S}\left(\mathrm{Sym}^2(y)\left(\left[(\sqrt{t},-\sqrt{t})\right]\right)\right) = \begin{cases} 4t & \text{if } t\geq 0 \\ \frac{1}{4}t & \text{if } t\leq 0 \end{cases}$$

<u>**Thm**</u> If Σ is a Riemann surface, then $\mathrm{Sym}^d\Sigma$ is a complex manifold.

<u>Proof.</u> Start from an atlas of Σ

$[(x_1,\dots,x_d)]\in \mathrm{Sym}^d\Sigma$ for x_i, consider $U_i\subseteq\Sigma$

s.t. $x_i = x_j \Rightarrow U_i = U_j$

$x_i \neq x_j \Rightarrow U_i\cap U_j = \emptyset$

neighbourhood of x can be identified with $\mathrm{Sym}^{d_1}(U_1)\times\dots\times \mathrm{Sym}^{d_m}(U_m)\cong\mathbb{C}^d$

Transition maps are symmetric products $\square$

Lemma $H_1(\mathrm{Sym}^d) \cong H_1(\Sigma)$

$\gamma \in H_1(\Sigma)$ s.t. $i_*(\gamma) = 0$

$$\begin{tikzcd} \Sigma \arrow[r, "i"] \arrow[dr, "\tilde{i}"'] & \mathrm{Sym}^d\Sigma \\ & \Sigma^d \arrow[u, "\pi"'] \end{tikzcd}$$

Fix $z \in \Sigma$ base pt

$i(x) = [(x, z, \dots, z)]$

$i_*: H_1(\Sigma) \to H_1(\mathrm{Sym}^d\Sigma)$

injective: take $\partial F \cong S^1$, $j: F \to \mathrm{Sym}^d\Sigma$

$j|_{\partial F} = i_*(\gamma)$

Make $j \pitchfork \Delta$ s.t. j lifts to $\tilde{j}: \tilde{F} \to \Sigma^d$

$\tilde{F}$ is a branched cover of F. Then we project $\tilde{j}$ to 1^{st} coordinate $\Rightarrow$ we have found some cycle with $\partial = \gamma$.

Inverse: $\gamma \in H_1(\mathrm{Sym}^d\Sigma)$ embedded, misses the diagonal $\Rightarrow$ lift it to $\tilde{\gamma} \subseteq \Sigma^d$ and project to 1^{st} coordinate

Thm $\pi_1(\mathrm{Sym}^d\Sigma) \cong H_1(\Sigma)$ $d \geq 2$

Lemma $X \xrightarrow{f} Y$ map of deg $d^{\neq 0}$ between closed, oriented mfds

Then $f^*: H^*(Y;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ is injective

Suppose $f^*(\alpha) = 0$ for some $\alpha \in H^*(Y;\mathbb{Q})$ $\alpha \neq 0$

Then $\exists\ \beta \in H^*(Y;\mathbb{Q})$ s.t. $\alpha \cup \beta = \omega_Y$ fund. class

$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta) = 0$ ↯ $\Rightarrow f^*$ injective. $\square$

$\| f^*\omega_Y = d \cdot \omega_X \neq 0$

Apply this to $\pi : \Sigma^d \to \mathrm{Sym}^d \Sigma$ $\quad \deg \pi = d! \neq 0$

$$H^*(\mathrm{Sym}^d \Sigma ; \mathbb{Q}) \xhookrightarrow{\pi^*} H^*(\Sigma^d ; \mathbb{Q})^{S_d}$$

classes fixed by action of permutations.

$H^1(\Sigma, \mathbb{Q})$ generated by $\alpha_1, \dots, \alpha_{2g}$

$H^2(\Sigma, \mathbb{Q})$ generated by ω

$H^*(\Sigma^d; \mathbb{Q})^{S_d}$ is generated as an algebra

by classes $\quad \xi_i = \sum_{j=1}^{d} 1 \otimes \dots \otimes 1 \otimes \underbrace{\alpha_i}_{j\text{:th pos}} \otimes 1 \otimes \dots \otimes 1$

$\in H^1(\mathrm{Sym}^d \Sigma, \mathbb{Q})$

$$\eta = \sum_{j=1}^{d} 1 \otimes \dots \otimes 1 \otimes \underbrace{\omega}_{j\text{:th pos}} \otimes 1 \otimes \dots \otimes 1 \in H^2(\mathrm{Sym}^d \Sigma; \mathbb{Q})$$

<u>Lemma</u> (direct computation): $\xi_1, \dots, \xi_{2g}, \eta$ generate $H^*(\Sigma^d; \mathbb{Q})^{S_d}$

To show that $H^*(\mathrm{Sym}^d \Sigma; \mathbb{Q}) \cong H^*(\Sigma^d; \mathbb{Q})^{S_d}$

we have to show that $\xi_1, \dots, \xi_{2g}, \eta$ are in the image of π^*

That ξ_i is in the image of π^* is a consequence of $H_1(\Sigma) \cong H_1(\mathrm{Sym}^d \Sigma)$ + rank argument

For η: claim $V_z = \{z\} \times \mathrm{Sym}^{d-1}\Sigma$

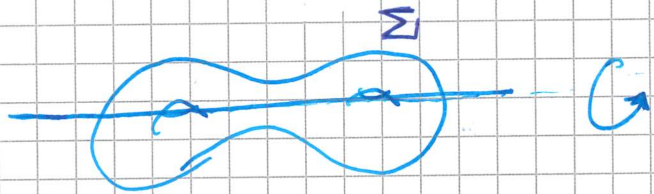
Poincaré dual of $V_z = \bar{\eta}$ Poincaré dual of

$$\pi^*(\bar{\eta}) = \pi^{-1}(V_z)$$

$$\pi^{-1}(V_z) = \text{P.D}(\eta)$$

$$\Rightarrow \pi^* \bar{\eta} = \eta$$

$\underline{d \geq 2}$ $\tau : \Sigma \to \Sigma$ hyperelliptic involution



$180°$ quotient $\cong S^2$

$z_3, \dots, z_d$ base pts $\neq z$

$$S = \{[(x, \tau(x), z_3, \dots, z_d)] \mid x \in \Sigma\} \subseteq \mathrm{Sym}^d \Sigma$$

$$\begin{matrix} \text{\rotatebox{90}{$\cong$}} \\ S^2 \end{matrix} \qquad V_z \cap S = \{[z, \tau(z), z_3, \dots, z_d]\}$$

<u>Prop</u> $[S]$ generates the image of $\pi_2(\mathrm{Sym}^d \Sigma) \to H_2(\mathrm{Sym}^d \Sigma; \mathbb{Q})$

<u>Proof</u> Any class in the image is $[S'] + k[S]$ with $S' \cdot V_z = 0 \Rightarrow \langle \eta, [S'] \rangle = 0$

$H^2(\mathrm{Sym}^d \Sigma; \mathbb{Q})$ generated by η, $\xi_i \cup \xi_j$

<u>Claim:</u> $\langle [S'], \xi_i \cup \xi_j \rangle = 0$

$$\begin{tikzcd} \pi_2(\widetilde{Sym}^d\Sigma) \arrow[r, "\cong", "p_*"'] \arrow[d, "\text{Hurewicz}"] & \pi_2(Sym^d\Sigma) \arrow[d, "\text{Hurewicz}"] \\ H_2(\widetilde{Sym}^d\Sigma) \arrow[r, "p_*"] & H_2(Sym^d\Sigma) \end{tikzcd}$$

univ. cover

univ. cover

$$[S'] = p_*[\tilde{S}'] \in H_2(\widetilde{Sym}^d\Sigma; \mathbb{Q})$$

$$\langle p_*[\tilde{S}'], \xi_i \cup \xi_j \rangle = \langle [\tilde{S}'], p^*\xi_i \cup p^*\xi_j \rangle = 0$$

Fix j complex str. on Σ s.t. z is holomorphic

S embedded, so $T\,Sym^d\Sigma|_S = TS \oplus N_S$

normal bundle

$$\langle c_1(T(Sym^d\Sigma)), [S] \rangle = \underset{2}{\underset{\|}{\chi(S)}} + \langle c_1(N_S), [S] \rangle$$

Call $\tilde{S} = \{(x, z(x), z_3, \dots, z_d)\} \subseteq \Sigma^d$

$x \in \tilde{S}$ $\quad d_x^N\pi : N_{\tilde{S}} \to N_S$ by quotient

$d_x\pi(T\tilde{S}) \subseteq TS$ $\quad$ this map is an iso except when $x \in \tilde{S} \cap \tilde{\Delta}$ and x not a branched pt of $\pi|_{\tilde{S}}$

These points are where x or $\tau(x) = z_3, \dots, z_d$

(we have $2(d-2)$ of them, coming in pairs $\{y, \tau(y)\}$)

<u>Claim</u> $N_{\tilde{S}} \cong \tau^* T\Sigma \oplus \underline{\mathbb{C}}^{d-2}$

$\langle c_1(N_{\tilde{S}}), [\tilde{S}]\rangle = 2-2g$

contributions from sing. of $d_x\pi$

$$\Rightarrow \langle c_1(N_S), [S]\rangle = \underbrace{\tfrac{1}{2}\langle c_1(N_{\tilde{S}}), \tilde{S}\rangle}_{\text{map } \pi \text{ is } 2:1} + \overbrace{d-2}$$

(use holomorph. + ker $(d_x\pi)$ is 1-dim)

$= 1-g+d-2 = g+d-1$

$$\Rightarrow \boxed{\langle c_1(T\mathrm{Sym}^d\Sigma), [S]\rangle = d-g+1}$$

$= 1$ when $d=g$