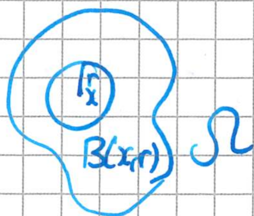
LIII

Cieliebak-Eliashberg: "From Weinstein to Stein & back"

Varouchas: "Stabilité de la classe des variétés Kähleriennes par certains morphismes propres"

Perutz: Hamiltonian handle slide for Heegaard Floer homology.

Plurisubharmonic functions



Def $\Omega \subseteq \mathbb{C}$ open set $\varphi \in C^0(\Omega)$ is subharmonic (sh) if $\forall x \in \Omega$, $\forall r > 0$ (suff small) and s.t. $B(x,r) \subseteq \Omega$ we have

$$\varphi(x) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(x + re^{i\theta})\, d\theta$$

Properties:

- Sh form a convex set.
- φ_1, φ_2 are sh, $\varphi := \sup(\varphi_1, \varphi_2)$ $\left(\varphi(x) = \max\{\varphi_1(x), \varphi_2(x)\} \forall x\right)$ is continuous & sh.
- sh satisfy the max. principle

  If $\exists x_0 \in \Omega$ s.t. $\varphi(x_0) \geq \varphi(x)$ $\forall x \in \Omega$ & Ω connected then φ is constant.
- If $\varphi \in C^\infty(\Omega)$, then φ is sh $\Longleftrightarrow$ $\Delta\varphi \geq 0$ $\forall x \in \Omega$

$\forall x \in \Omega, \quad \lambda_x(r) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(x + re^{i\theta}) d\theta$

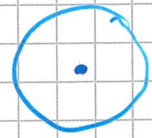
$\Rightarrow \lambda_x$ has a local min. in $r=0 \Rightarrow \lambda_x''(0) \geq 0$

$\lambda_x'(r) = \frac{1}{2\pi r} \int_{B(x,r)} \Delta \varphi(x + re^{i\theta}) \leftarrow$ differentiate λ & apply Gauß-Green

$\lambda_x''(0) = 2 \Delta \varphi(x)$

<u>Def</u> $\varphi \in C^0(\Omega)$ is <u>harmonic</u> if φ & $-\varphi$ are subharmonic.

<u>Prop</u>
- vector space
- satisfy the mean value thm.
- in $C^\infty(\Omega)$ and $\Delta \varphi = 0$



<u>Dfn</u> $\Omega \subseteq \mathbb{C}^n$ open set, $\varphi \in C^0(\Omega)$ (real valued) is plurisubharmonic (psh) if for all holomorphic maps $u: B(0,R) \to \Omega$ the composition $\varphi \circ u$ is sh.
$\quad\quad B(0,R) \subseteq \mathbb{C}$

<u>Properties</u>

- convex set
- φ_1, φ_2 are psh on $\Omega \Rightarrow \sup(\varphi_1, \varphi_2)$ is psh on Ω
- $\psi: \mathbb{C}^m \to \mathbb{C}^n$ is holomorphic $\varphi \in C^0(\Omega)$, $\Omega \in \mathbb{C}^n$ is psh then $\varphi \circ \psi$ is psh.

- maximum principle

$\varphi \in C^\infty(\Omega)$, $\quad d^c\varphi = d\varphi \circ J$

J complex str. on $\mathbb{C}^n$

__Prop__ $\varphi \in C^\infty(\Omega)$ is psh iff $-dd^c\varphi \geq 0$ on the J-complex lines in Ω (w. complex orientation)

__Proof__ w.l.o.g. assume that $\Omega \subseteq \mathbb{C}$ and φ is sh.

coordinate $z = x + iy$

$d\varphi = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy$

$J\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}$, $J\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x}$

$dx \circ J = -dy$, $dy \circ J = dx$

$d^c\varphi = d\varphi \circ J =$

$$= -\frac{\partial \varphi}{\partial x} dy + \frac{\partial \varphi}{\partial y} dx$$

$$dd^c\varphi = -\frac{\partial^2 \varphi}{\partial x^2} dx \wedge dy + \frac{\partial^2 \varphi}{\partial y^2} dy \wedge dx$$

$\Rightarrow -dd^c\varphi = \Delta\varphi \boxed{dx \wedge dy}$ complex orient.

φ is sh iff $\Delta\varphi \geq 0$

__Dfn__ $\varphi \in C^\infty(\Omega)$ is strictly psh (spsh) if $-dd^c\varphi > 0$

$\varphi \in C^0(\Omega)$ is spsh if $\varphi = \varphi_0 + \varphi_1$, $\varphi_1 \in C^\infty(\Omega)$ spsh, $\varphi_0 \in C^0(\Omega)$ psh

Rmk $\varphi \in C^\infty(\Omega)$ is spsh iff $-dd^c\varphi$ is a symplectic form on Ω which is compatible with the std complex str.

$z_j = x_j + i y_j$, $j = 1, \dots, n$ coordinate in $\mathbb{C}^n$

$$-dd^c\varphi = \sum_{j=1}^{n} \left(\frac{\partial^2 \varphi}{\partial x_j^2} + \frac{\partial^2 \varphi}{\partial y_j^2} \right) dx_j \wedge dy_j$$

and $\displaystyle \frac{\partial \varphi^2}{\partial x_j^2} + \frac{\partial \varphi^2}{\partial y_j^2} > 0$

[ω sympl. form
J a.c.s. is compatible if
$\omega(\cdot, J\cdot)$ is a Riemannian metric)]

T measured space w. a finite measure μ

$\Omega \subseteq \mathbb{C}^n$, $\varphi : T \times \Omega \to \mathbb{R}$ s.t.

- $\forall t \in T$, $z \mapsto \varphi(t, z)$ is continuous & psh
- $\forall x \in \Omega$, $t \mapsto \varphi(t, z)$ is measurable. Then

$\psi(x) = \int_T \varphi(t, x) d\mu(t)$ is cont & psh.

Moreover, if $\varphi = \varphi_0 + \varphi_1$ s.t. $\varphi_0 : T \times \Omega \to \mathbb{R}$ as above & φ_1 satisfies:

• $\forall t \in T$, $x \mapsto \varphi(t,x)$ smooth and for $\forall t$ outside zero measure set s.p.sh

• $\forall x \in \Omega$, $t \mapsto \varphi(t,x)$ is measurable

Then ψ is spsh.

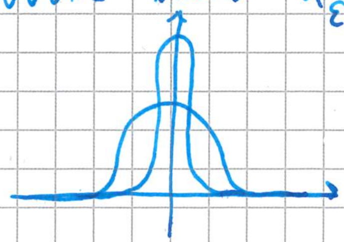
Exercise compute $c_1(T\,\mathrm{Sym}^4\Sigma)$ or $c_1(T^*\mathrm{Sym}^4\Sigma)$

$\alpha : \underset{\subseteq \mathbb{C}^n}{B(0,1)} \to \mathbb{R}$ compactly supp. smooth funct. s.t. $\int_{B(0,1)} \alpha = 1$

$\alpha(x) \geq 0$

$\alpha_\varepsilon(x) = \varepsilon^{-n}\alpha\left(\frac{x}{\varepsilon}\right)$ Note that α_ε has supp in $B(0,\varepsilon) \subseteq \mathbb{C}^n$

and $\int_{B(0,\varepsilon)} \alpha_\varepsilon = 1$



$\varphi : \Omega \to \mathbb{R}$ (s) psh

$\varphi_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ Ω_ε : points in Ω at distance $> \varepsilon$ from $\mathbb{C}^n \setminus \Omega$

$$\varphi_\varepsilon(x) = (\varphi * \alpha_\varepsilon)(x) = \int_{B(0,\varepsilon)} \varphi(x+x')\alpha_\varepsilon(x')\,dx'$$

Properties : • φ is cont. and (s)psh, then φ_ε is smooth and (s)psh

• $\varphi_\varepsilon \xrightarrow{C^0} \varphi$ as $\varepsilon \to 0$.

Fix $\alpha: \mathbb{R}^2 \to \mathbb{R}$ smooth s.t.

- α has support in $(-1,1)^2$
- $\alpha(s_1, s_2) = \alpha(-s_1, -s_2)$
- $\iint \alpha(s_1, s_2)\, ds_1\, ds_2 = 1$

$$\alpha_\varepsilon(s_1, s_2) = \varepsilon^{-n} \alpha\left(\frac{s_1}{\varepsilon}, \frac{s_2}{\varepsilon}\right)$$

For $\delta > 0$ we define

$$M_\delta(s_1, s_2) := \iint \sup(s_1 - \delta t_1, s_2 - \delta t_2)\, \alpha(t_1, t_2)\, dt_1\, dt_2$$

- M_δ is the convolution of $(s_1, s_2) \mapsto \sup(s_1, s_2)$ with α_δ.

Lemma $M_\delta(s_1, s_2) = \sup(s_1, s_2)$ if $|s_1 - s_2| \geq 2\delta$

Suppose $s_1 - s_2 \geq 2\delta$. Then for $|t_1|, |t_2| < 1$ we have $s_1 - \delta t_1 > s_2 - \delta t_2$. Then $\sup(s_1 - \delta t_1, s_2 - \delta t_2)$
$= s_1 - \delta t_1$

$$M_\delta(s_1, s_2) = \iint (s_1 - \delta t_1)\, \alpha(t_1, t_2)\, dt_1\, dt_2 = s_1 - \delta \underbrace{\iint t_1 \alpha(t_1, t_2)\, dt_1\, dt_2}_{=0 \text{ (odd fcn symm. int.)}}$$

$$= s_1$$

Lemme If $\varphi_1, \varphi_2 \in C^0(\Omega)$ and (s)psh, then

- $M_\delta(\varphi_1, \varphi_2)$ is (s)psh and φ_1, φ_2 are smooth on $\Omega' \subseteq \Omega$ **then** $M_\delta(\varphi_1, \varphi_2)$ is smooth on Ω'.

closure is included

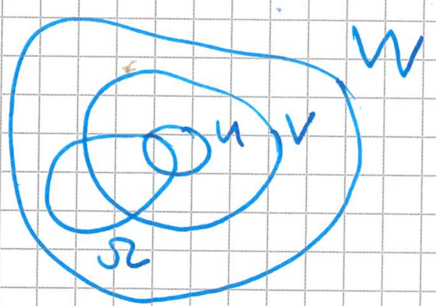
Lemme principal. $U \Subset V \Subset W \subset \mathbb{C}$ open sets

$\Omega \subseteq W$ open.

$\varphi \in C^0(W)$ spsh, $\varphi|_\Omega$ is smooth.

Then $\exists\ \psi \in C^0(W)$ spsh, $\psi|_{\Omega \cup U}$ smooth,

$$\psi|_{W \setminus \overline{V}} = \varphi|_{W \setminus \overline{V}}$$

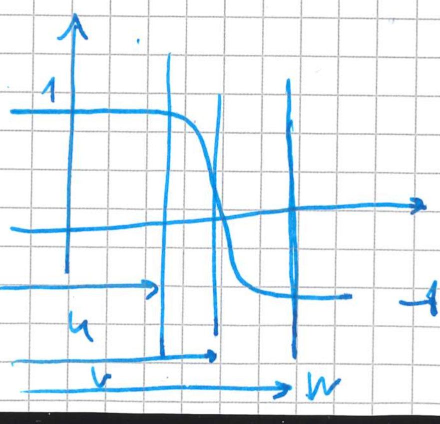


$\alpha_\varepsilon \rightsquigarrow \varphi_\varepsilon = \varphi * \alpha_\varepsilon \xrightarrow{C^0} \varphi$ smooth and spsh

For ε small enough, φ_ε is defined on $\overline{V}$

$\eta : W \to [-1, 1]$ smooth & s.t.

$\eta|_U \equiv 1$ and $\eta|_{W \setminus V} \equiv -1$



Find t, ε small enough s.t.

$|\varphi_\varepsilon - \varphi| < \frac{t}{2}$ on $\bar{V}$

$\varphi_\varepsilon + t\eta$ is spsh on a nbhd of $\bar{V}$ W'

Fix $\delta < \frac{t}{4}$ and define $\psi_\delta = M_\delta(\varphi, \varphi_\varepsilon + t\eta)$

- ψ_δ is smooth on $\Omega \cap W'$

- $\psi_\delta|_U = \varphi_\varepsilon + t\eta$ so is smooth and spsh

  $\psi_\delta|_{W' \setminus V} = \varphi|_{W' \setminus V}$

<u>Define</u> $\psi(x) = \begin{cases} \psi_\delta(x) & \text{if } x \in \bar{V} \\ \psi(x) & \text{if } x \in W \setminus V. \end{cases}$

<u>Dfn</u> $\varphi \in C^0(\Omega)$ is pluriharmonic if $\varphi, -\varphi$ are psh

$\Rightarrow$ φ smooth and $dd^c \varphi = 0$

<u>Dfn</u> X complex mfd, $\{U_i\}_{i \in I}$ loc. finite cover by coordinate charts (i.e. "$U_i \overset{open}{\subseteq} \mathbb{C}^n$")
and a family $\varphi_i : U_i \to \mathbb{R}$ spsh s.t.
$\forall i, j$ $\varphi_i - \varphi_j$ pluriharmonic, def. on $U_i \cap U_j$.

<u>Remark</u> ($\{(U_i, \varphi_i)\}$ smooth K.C. then
(all φ_i smooth))

$\omega_i = -dd^c \varphi_i$ is symplectic on U_i and comp. w/ J.

$\omega_i - \omega_j = -dd^c(\varphi_i - \varphi_j) = 0 \Rightarrow \exists$ sympl. form ω on X

$\omega|_{U_i} = \omega_i$ comp. w. J

<u>Prop</u> X complex mfd, $X = X_0 \cup X_1$

$\{(W_i, \varphi_i)\}$ is C^0 K.C. subordinated to X_0, X_1:

$\left(I = I_0 \amalg I_1, \quad U_i \subseteq X_* \text{ if } i \in I_* \text{ for } * = 0,1 \right)$

and φ_i smooth if $i \in I_0$ $(\Rightarrow U_i \subseteq X_0)$

by inductively applying lemme principal to

$(U_i, \varphi_i)_{i \in I_1}$ we get a smooth K.C. $\{(W_i, \psi_i)\}$

and $\xi : X \longrightarrow \mathbb{R}$ supp $\xi \subseteq X_1$ s.t.

$$\psi_i = \varphi_i + \xi|_{W_i} \quad \forall i \in I_1.$$

<u>Thm</u> Fix ω an area form on Σ. Then $\exists$

Ω symplectic form on $\mathrm{Sym}^d \Sigma$ s.t.

- outside of a nbhd of Δ, $\Omega = \pi_* \omega^{\times d}$, $\omega^{\times d}$ on Σ^d ($\pi_{1*}\omega \oplus \pi_{2*}\omega \oplus \ldots$)
- Ω exact on $\mathrm{Sym}^d \Sigma \setminus V_z$ $\left(\begin{array}{l} z \in \Sigma \\ V_z \cong \{z\} \times \mathrm{Sym}^{d-1} \Sigma \end{array} \right)$

$(\Sigma, \omega) \rightsquigarrow \left(\{ (A_i, \bar{\varphi}_i)_{i \in I} \} \right)$ K.C.

$$A_i \subseteq \Sigma, \quad \omega|_{A_i} = f_i \, dx_i \wedge dy_i$$

$$-dd^c \bar{\varphi}_i = \Delta \bar{\varphi}_i \, dx_i \wedge dy_i$$

We are looking for sol of $\Delta \bar{\varphi}_i = f_i$

This will produce a Kähler cocycle on Σ^d

$\rightsquigarrow$ K.C. on $\mathrm{Sym}^d \Sigma$: $\forall x \in \mathrm{Sym}^d \Sigma$

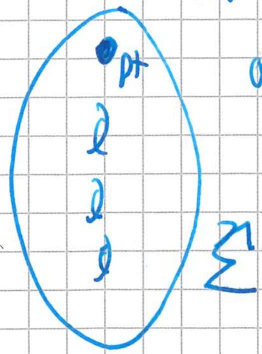
find a suff small nbhd. U, $\varphi : U \to \mathbb{R}$ st

$[(x_1, \dots, x_d)] \in U \Rightarrow \varphi_U([(x_1, \dots, x_d)])$

$$= \bar{\varphi}_1(x_1) + \dots + \bar{\varphi}_d(x_d)$$

$-dd^c \bar{\varphi} = \omega$ sympl.

outside $pt \in \Sigma$



<u>Exercise</u> $(\mathbb{C}, \bar{\varphi})$, $\bar{\varphi}$ spsh e.g. x^2, $\frac{1}{2}|z|^2$

do the construction for $\mathrm{Sym}^2 \mathbb{C} \cong \mathbb{C}^2$