L IV

1. Heegaard diagrams

Y closed, connected, oriented 3-mfd

$Y \to \mathbb{R}$ Morse

- unique max and min
- g index 1 & index 2 crit pts $g>0$
- $x \in Y$ crit pt of f then

$$\begin{cases} f(x)>0 & \text{if} \quad ind(x) = 2,3 \\ f(x)<0 & \text{if} \quad ind(x) = 0,1 \end{cases}$$

$\Sigma = f^{-1}(0)$ Heegaard surface

$U_\alpha = f^{-1}((-\infty, 0])$, $U_\beta = f^{-1}([0, +\infty))$ handle bodies

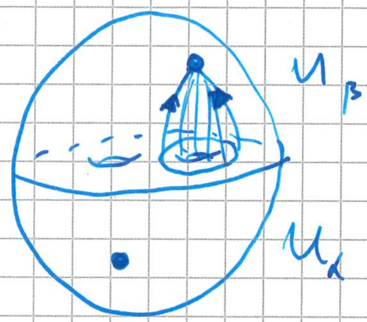
$Y = U_\alpha \cup_\Sigma U_\beta$, orient $\Sigma = \partial U_\alpha = -\partial U_\beta$

Heegaard splitting, Σ Heegaard surface

Chose a Morse-Smale pseudogradient for f

$\alpha_1, \dots, \alpha_g$ = intersections between Σ & unstable mfds of index 1 crit. pts.

$\beta_1, \dots, \beta_g$ = — " — Σ & stable mfds of index 2 crit pts.



$\underline{\alpha} = (\alpha_1, \dots, \alpha_g)$, $\underline{\beta} = (\beta_1, \dots, \beta_g)$

$(\Sigma, \underline{\alpha}, \underline{\beta})$ Heegaard diagram

choose $z \in \Sigma \setminus (\underline{\alpha} \cup \underline{\beta})$

$(\Sigma, \underline{\alpha}, \beta, z)$ pointed Heegaard diagram

Fix ω area form, j complex str. on Σ:

$\rightsquigarrow \mathrm{Sym}^g \Sigma$, Ω sympl, $J = \mathrm{Sym}(j)$ comp. w. Ω

$\mathbb{T}_\alpha = \alpha_1 \times \dots \times \alpha_g \qquad \mathbb{T}_\beta = \beta_1 \times \dots \times \beta_g$

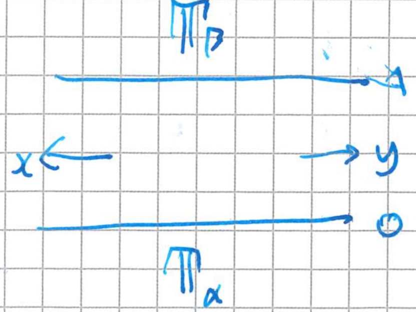
$\Omega\big|_{T\mathbb{T}_\alpha} = \Omega\big|_{T\mathbb{T}_\beta} = 0 \qquad \mathbb{T}_\alpha, \mathbb{T}_\beta \cap \Delta = \emptyset$

$\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is $\mathbf{x} = (x_1, \dots, x_g)$ where x_i is an int. pt. between an α-curve and a β-curve

each α, β curve counted only once

$x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \qquad \mathcal{B}(x,y) = \{$ smooth $u : \mathbb{R} \times [0,1] \to \mathrm{Sym}^g \Sigma$

s.t.
- $u(\mathbb{R} \times \{0\}) \subseteq \mathbb{T}_\alpha$
- $u(\mathbb{R} \times \{1\}) \subseteq \mathbb{T}_\beta$
- $\lim_{s \to -\infty} u(s,t) = x$
- $\lim_{s \to +\infty} u(s,t) = y$



"Whitney discs"

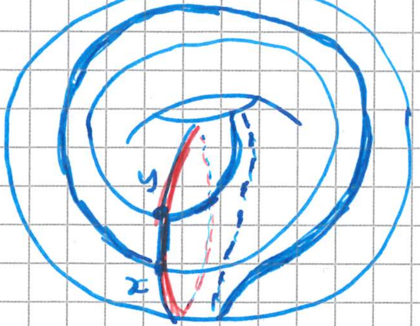
$x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ we choose a path

$a = (a_1, \dots, a_g)$ $\quad a_i \subseteq \alpha_i$ from x to y,

$b = (b_1, \dots, b_g)$, $\quad b_i \subseteq \beta_i$ from x to y

Concatenate a and $-b \rightsquigarrow \varepsilon(x,y)$ loop in Σ

its homology class in $H_1(Y)$ is well def.



Prop If $\varepsilon(x,y) \neq 0$ then $\mathcal{B}(x,y) = \emptyset$

Sketch suppose $u \in B(x,y)$, make it $\pitchfork$ to Δ

we can lift it to

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[rr, bend right] & \Sigma^g \arrow[r, "proj"] & \Sigma \\ \mathbb{R}\times[0,1] & & \end{tikzcd}$$

$\tilde{u} : F \to \Sigma$

s.t. $\tilde{u}(\partial F) = \varepsilon(x,y)$

$\Rightarrow \varepsilon(x,y) = 0$

$x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ are spin^c-equivalent if $\varepsilon(x,y) = 0$

$\Sigma \setminus (\underline{\alpha} \cup \underline{\beta}) = D_0 \cup \dots \cup D_m$ on D_i choose $z_i \in D_{z_i}$

(s.t. $z = z_0$)

$V_{z_i} \cong \{z_i\} \times \mathrm{Sym}^{g-1}\Sigma$

Given $u \in \mathcal{B}(x,y)$, $\quad n_{z_i}(u) = \# u^{-1}(V_{z_i})$

(after making transverse)

$u \in \mathcal{B}(x,y) \rightsquigarrow \mathcal{D}(u) = \sum_{i=0}^{m} n_{z_i}(u)\, D_i$

If $u, v \in \mathcal{B}(x,y)$ are homotopic $\Rightarrow \mathcal{D}(u) = \mathcal{D}(v)$

(converse is true if $g \neq 2$.)

Define $\pi_2(x,y) = \mathcal{B}(x,y) \big/ \begin{smallmatrix} u \sim v \\ \text{if } \mathcal{D}(u) = \mathcal{D}(v) \end{smallmatrix}$

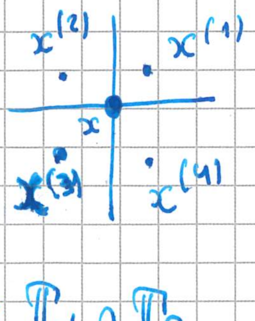
$\pi_0 \mathcal{B}(x,y) \to \pi_2(x,y)$ well-def map

Define $e(D_i) = \chi(D_i) - \frac{\#\text{vertices}}{4}$ $\quad$ <u>Euler measure</u>

$\phi \in \pi_2(x,y) \quad \mathcal{D}(\phi) = \sum a_i D_i$

$a_i = n_{z_i}(\phi)$

$x \in \alpha_i \cap \beta_j$



$\underline{n}_x(D_i) = \frac{1}{4}\left(n_{x^{(1)}}(D_i) + \dots + n_{x^{(4)}}(D_i)\right)$

$\mathbf{x} = (x_1, \dots, x_g) \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$

$\underline{n}_{\mathbf{x}}(D_i) = \underline{n}_{x_1}(D_i) + \dots + \underline{n}_{x_g}(D_i) \quad \underline{n}_{\mathbf{x}}(\phi) = \sum a_i\, \underline{n}_{\mathbf{x}}(D_i)$

$\mu : \pi_2(x,y) \to \mathbb{Z}$ Maslov index

$\mu(\phi) = e(\phi) + n_x(\phi) + n_y(\phi)$ (Lipschitz formula)

$x_+, x_0, x_- \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$

$$\pi_2(x_-, x_0) \times \pi_2(x_0, x_+) \to \pi_2(x_-, x_+)$$

$$\phi_1 \qquad \phi_2 \longmapsto \phi_1 * \phi_2$$

$$D(\phi_1 * \phi_2) = D(\phi_1) + D(\phi_2)$$

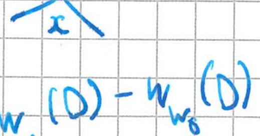
$$e(\phi_1 * \phi_2) = e(\phi_1) + e(\phi_2)$$

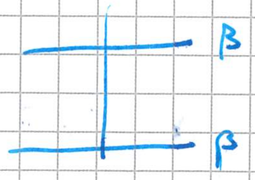
Prop [Sarkar] $\mu(\phi_1 * \phi_2) = \mu(\phi_1) + \mu(\phi_2)$

Proof. $\mu(\phi_1 * \phi_2) - \mu(\phi_-) - \mu(\phi_+) =$

$$\underline{n_{x_-}(\phi_+) + n_{x_+}(\phi_-) - n_{x_0}(\phi_+) - n_{x_0}(\phi_-)} \overset{?}{=} 0$$

$\partial\phi_+ \circ \partial\phi_- = 0$



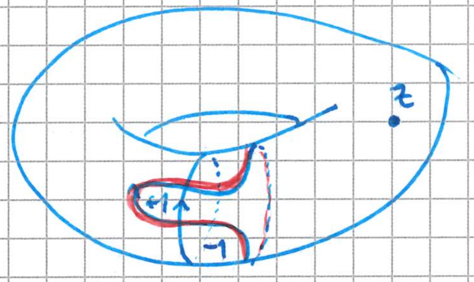


$n_{w_1}(D) - n_{w_0}(D) = \gamma \cdot \partial D$

$\square$

$P = \sum a_i D_i$ is <u>periodic domain</u> if $a_0 = 0$ ($n_z(P) = 0$)

and if ∂P is a linear comb. of α & β curves



Exact sequence for (Y, Σ):

kernel $= \Pi$ space of periodic domains

$$0 \to H_2(Y) \to H_1(\underline{\alpha}) \oplus H_1(\underline{\beta}) \to H_1(\Sigma) \to H_1(Y) \to 0$$

This gives an identification $\Pi \cong H_2(Y)$

$$\begin{array}{ccc} \Pi & \cong & H_2(Y) \\ \Downarrow & & \Downarrow \\ P & \mapsto & \hat{P} \end{array}$$ homology class

<u>Lemma</u> $\pi_2(x, x) \cong \underset{\substack{\Downarrow \\ n_z(\phi)}}{\mathbb{Z}} \oplus \Pi$

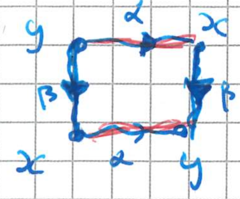
restrict $\mu : \pi_2(x, x) \to \mathbb{Z}$ to Π, this gives a linear map $c_1(x) : H_2(Y) \to \mathbb{Z}$

We constructed a map

$c_1 : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to H^2(Y)$

$\langle c_1(x), \hat{P} \rangle = \chi(P) + \underline{2} n_x(P)$

<u>even</u>



Lemma $c_1(y) - c_1(x) = 2\,PD(\varepsilon(x,y))$

Proof $P \in \Pi$, $\langle c_1(y), \hat{P} \rangle - \langle c_1(x), \hat{P} \rangle$

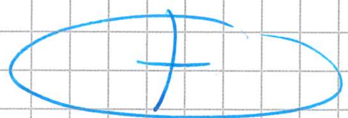
$$= 2\left(n_y(P) - n_x(P)\right)$$

$\varepsilon_\alpha(x,y)$, $\varepsilon_\beta(x,y)$

$\subseteq \alpha$ $\qquad \subseteq \beta$

Then $n_y(P) - n_x(P) = \varepsilon_\beta(x,y) \circ \partial_\alpha P$

$\hat{P} = P \cup \alpha$- & β-discs

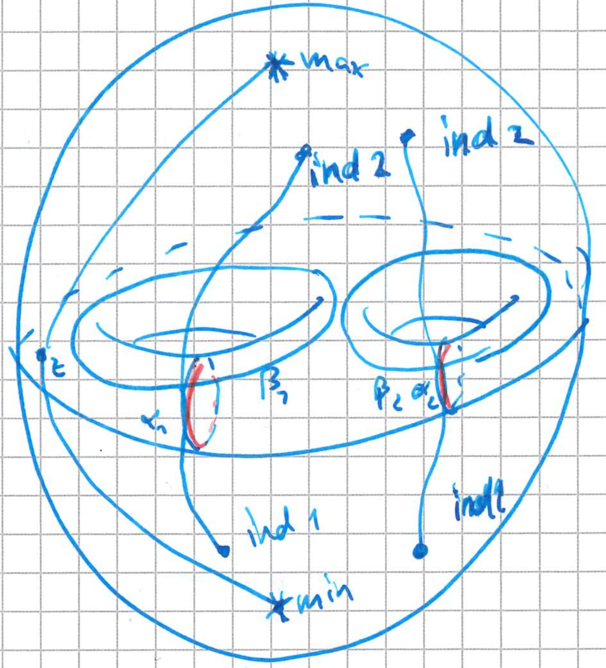


ε_β pushed down

$$= 2\,\varepsilon(x,y) \cdot \hat{P}$$

Y 3-mfd. A spin^c-structure on Y is a nowhere vanishing vector field (plane field) up to homotopy outside a ball ($\Leftrightarrow$ outside finite nr of balls)

$x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow s(x) \in Spin^c(Y)$



Remove small balls around gradient trajectories, we can remove the sing. of the gradient v.f. there by a homotopy.

$Spin^c(Y)$ is an affine space over $H^2(Y) \cong H_1(Y)$

$$s(y) - s(x) = Pd(\varepsilon(x,y))$$

(Removing balls in 3-mfd does not change $H_2(Y^3)$)