LV

$Y \rightsquigarrow (\Sigma, \alpha, \beta, z)$ pted Heegaard diagram

Choose ω area form on Σ

j_t, $t \in [0,1]$ a path of complex str on Σ

$\rightsquigarrow$ $Sym^g \Sigma$, Ω symplectic form, admissible path

$J_t = Sym^g_{j_t}$ a.c.s. on $Sym^g \Sigma$ (integrable)

If j_t is sufficiently close to a constant path

Ω is tamed by J_t for all $t \in [0,1]$

$\Omega(\cdot, J_0 \cdot)$ Riemannian metric

$\Omega(\cdot, J_t \cdot) > 0$ for $t \in [0,1]$

Given $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\phi \in \pi_2(x,y)$

We define $\mathcal{M}(\phi) = \{ u \in \phi \mid \frac{\partial u}{\partial s} + J_t \frac{\partial u}{\partial t} = 0 \}$

CR - equation

$u \in \mathcal{M}(\phi)$, $u(\cdot + s_0, \cdot) \in \mathcal{M}(\phi) \Rightarrow \mathbb{R}$ acts on $\mathcal{M}(\phi)$

call $\widehat{\mathcal{M}}(\phi) = \mathcal{M}(\phi)/\mathbb{R}$

Recall $z_i \in D_i$, $\Sigma \setminus (\underline{\alpha} \cup \underline{\beta}) = D_0 \sqcup \dots \sqcup D_n$

Require that j_t is constant on small nbhds of z_i

<u>Lemma</u> If $\mathcal{M}(\phi) \neq \emptyset$ then $\mathcal{D}(\phi) > 0$

i.e. if $u \in \mathcal{M}(\phi)$ then $n_{z_i}(u) \geq 0$ $\forall z_i$

<u>Proof</u> (take $z_i = z$) take $\underline{w} \in \mathrm{Im}(u) \cap V_z$

$$\underline{w} = (z, \dots, z, w_{k+1}, \dots, w_g) \qquad w_i \neq z$$

W neigh of w of the form $\mathrm{Sym}^k(U) \times U_{k+1} \times \dots \times U_g$

U neigh. of z on which j_t cst.

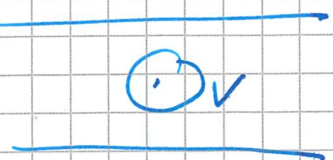
U_i is a neigh. of w_i for $i = k \dots g$

$f : U \to \mathbb{C}$ holom. fcn s.t. $f^{-1}(0) = z$ $d_0 f \neq 0$

$x = (x_1, \dots, x_g) \in \mathcal{U}$ define $F(x) = f(x_1) \dots f(x_k)$

~~We can see that mult. of w as an intersection~~

(s_0, t_0) s.t. $u(s_0, t_0) = w$



$F \circ u|_V$ and (s_0, t_0) is a zero, the mult of w in $\mathrm{Im}\, u \cap V_z$ = mult of (s_0, t_0) as zero of $F \circ u|_V > 0$

<u>Cor</u> If $n_{z_i}(u) = 0$ then $\mathrm{Im}\, u \cap V_{z_i} = \emptyset$.

Thm For a generic admissible path J_t

$$\left(\text{admissible} \Rightarrow J_t = \mathrm{Sym}^g(j_t),\ \underbrace{\text{sufficiently constant}}_{\text{sympl. form constr. only so still tames all } j_t} \right.$$

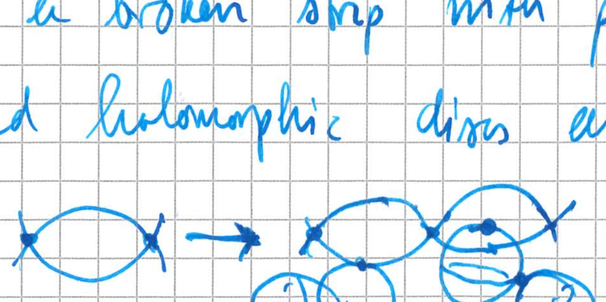
$$\left. \text{constant on } z_0, \dots, z_1 \right)$$

The moduli space $\mathcal{M}(\phi)$ for $\phi \in \pi_2(x,y)$ is

- if $\mu(\phi) < 0$, $\mathcal{M}(\phi) = \emptyset$
- if $\mu(\phi) = 0$, $u \in \mathcal{M}(\phi)$ is constant ($\Rightarrow x = y$)
- if $\mu(\phi) > 0$, then $\widehat{\mathcal{M}}(\phi)$ is a smooth manifold of $\dim \widehat{\mathcal{M}}(\phi) = \mu(\phi) - 1$

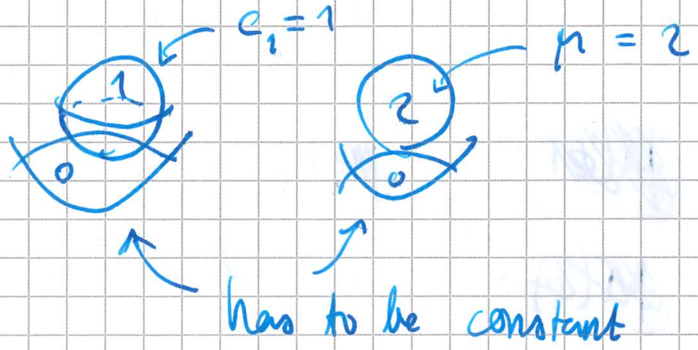
Thm (Gromov-Floer compactness)

If $J_t^{(n)} \to J_t^{(\infty)}$ is a sequence of admissible paths and u_n a sequence of $J_t^{(n)}$-holomorphic strips in $\mathcal{M}(\phi_n)$ for $\phi_n \in \pi_2(x,y)$ with $\int u_n^* \Omega < C$, then up to a subsequence u_n converges to a broken strip with possibly holomorphic spheres and holomorphic discs attached:



<u>However</u>: If $J_t^{(\infty)}$ is generic, then three possibilities (by additivity of index)

If $\mu(\Phi_n) = 2$



<u>Lemma</u> Bubbling of spheres does not occur

$c_1 = 1$ spheres live in a smooth space of $2(g-3)+2$
$= 2g-4$

$\Rightarrow$ image of spheres of $c_1 = 1$ trace out a codim-2 subset

Generically $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ misses all hol. sphere w. $c_1 = 1$

$x \in \mathbb{T}_\alpha$, $\mathcal{M}_\alpha(x) = \{ u : D^2 \to \mathrm{Sym}\,\Sigma$ s.t. $\bar{\partial}_{J_0} u = 0$

$u(\partial D^2) \subseteq \mathbb{T}_\alpha$, $[u] = [S]$

(i.e. $\mu(u) = 2$, $n_z(u) = 1$

bdy of u passes through $x \}$

dim $\mathcal{M}_\alpha(x)$ to be a finite nr of pts

Claim $\mathcal{M}_\alpha(x)$ transversely cut out for generic J_0 because $n_z(u) = 1$, and compact because of min energy.

Prop $\# \mathcal{M}_\alpha(x) \neq 0$

Proof ① $\# \mathcal{M}_\alpha(x)$ independent of the (generic) symmetric complex str.

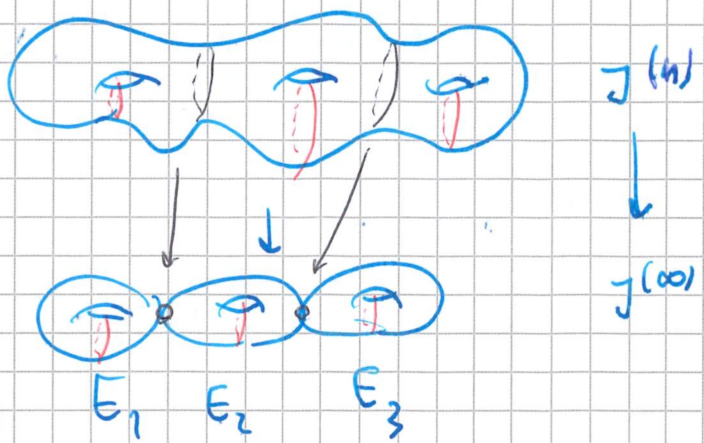
② degenerate $\Sigma \to E_1 \vee \dots \vee E_g$ nodal

all $E_i \cong T^2$, $\alpha_i \subseteq E_i$

$u_n \in \mathcal{M}_\alpha(x, J^{(n)})$

$\downarrow$

$u^{(\infty)}$ nodal disc



in $\mathrm{Sym}^g(E_1 \vee \dots \vee E_g)$

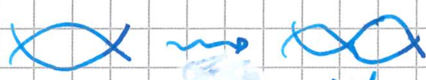
$$\mathrm{Sym}^g(E_1 \vee \dots \vee E_g) = \coprod \mathrm{Sym}^{k_1}(E_1) \times \dots \times \mathrm{Sym}^{k_g}(E_g)$$

where $k_i \geq 0$, $k_1 + \dots + k_g = g$

$u^{(\infty)}$ consists of a disc in $E_1 \times \dots \times E_g$ with boundary on $\alpha_1 \times \dots \times \alpha_g$ + (possibly) spheres in the other irr. components.

$\pi_2(E_1 \times \dots \times E_g, \alpha_1 \times \dots \times \alpha_g) = 0$, so this disc is constant

breakings from now on



only

$\widehat{CF}(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ is the vector space (over $\mathbb{F}_2$) generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Define $\hat{\partial}: \widehat{CF} \to \widehat{CF}$ by:

$$\hat{\partial} x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1 \\ n_z = 0}} \# \widehat{\mathcal{M}}(\phi)\, y$$

$CF^\infty(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ as the vector space generated by pairs $[x, i]$ s.t. $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $i \in \mathbb{Z}$

$$\partial^\infty [x, i] = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \# \widehat{\mathcal{M}}(\phi)\, [y, i - n_z(\phi)]$$

$n_z(\phi) \geq 0$ if $\widehat{\mathcal{M}}(\phi) \neq \emptyset$ implies

$CF^-(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ subcomplex of CF^∞ gen. by $[x, i]$ where $i < 0$.

$$CF^+(\Sigma, \underline{\alpha}, \underline{\beta}, z) = \frac{CF^\infty(\Sigma, \underline{\alpha}, \underline{\beta}, z)}{CF^-(\Sigma, \underline{\alpha}, \underline{\beta}, z)}$$

$CF^+(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ is generated by $[x,i]$ with $i \geq 0$

$$\partial^+[x,i] = \sum_{y\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{j=0}^{i}\ \sum_{\substack{\phi\in\pi_2(x,y)\\ \mu(\phi)=1\\ n_z(\phi)=j}} \#\widehat{\mathcal{M}}(\phi)[y,i-j]$$

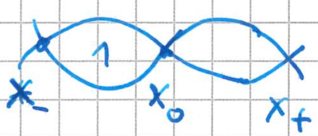
$$U: CF^\infty(\Sigma, \underline{\alpha}, \underline{\beta}, z) \to CF^\infty(\Sigma, \underline{\alpha}, \underline{\beta}, z)$$

$U[x,i] = [x,i-1]$ $\quad$ U is a chain map

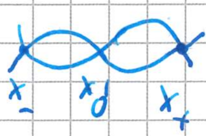
<u>Exercise</u> Prove the tautological exact sequences

<u>Why complexes?</u> $\quad x_\pm \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$

$\langle \hat{\partial}^2(x_-), x_+\rangle$ counts broken strips



(for all intersections x_0)



Count gives $=0$ since broken strips come in pairs.

<u>Dfn</u> $(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ is <u>weakly admissible</u>

for a Spin^c-structure s if for all periodic

domains P w. $\langle c_1(s), \hat{P}\rangle = 0$, P has

positive & negative coefficients.

<u>Dfn</u> $(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ is <u>strongly admissible</u> if

for s if for all periodic domain P s.t.

$\langle c_1(s), \hat{P}\rangle = 2n \geq 0$, then P has a coefficient $> n$.

<u>Prop</u> If $(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ is weakly admissible for all

$i \geq 0$, $j \in \mathbb{Z}$, $\exists$ finitely many $\phi \in \pi_2(x,y)$ s.t.

$\mu(\phi) = j$, $n_z(\phi) = i$, $\mathcal{D}(\phi) \geq 0$

Enough for $\widehat{CF}$, CF^+ $\Big\}$ <u>only nonzero for finitely many Spin^c</u>

If strongly admissible, $\forall j \in \mathbb{Z}$ $\exists$ finitely many

$\phi \in \pi_2(x,y)$ s.t. $\mu(\phi) = j$, $\mathcal{D}(\phi) \geq 0$,

enough for CF^∞ and CF^- $\Big\}$ <u>always</u> non-trivial

If $\exists\ \theta \in \Omega^1(\Sigma \setminus \{z\})$ s.t. $d\theta$ is an area form and $\int_{\alpha_i} \theta = \int_{\beta_i} \theta = 0$ for all i then

$(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ is weakly admissible for all spin^c-str.

$$\int_P d\theta = \int_{\partial P} \theta = 0, \quad P = \sum a_i D_i \quad \int_P d\theta = \sum a_i \underbrace{\int_{D_i} d\theta}_{>0}$$

This implies that $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are exact for some choice of primitive of Ω

<u>Lemma</u> $\eta \in \Omega^1_c(\Sigma \setminus \{z\})$ s.t. $d\eta = 0$, then

X_η vector field $\iota_{X_\eta} d\theta = \eta$, φ_t flow of X_η

Then for all $\gamma \in \Sigma$ s.c.c. we have $\int_{\varphi_t(\gamma)} \theta = \int_\gamma \theta + t\eta$

θ any primitive of an area form

choose $\eta \in \Omega^1_c(\Sigma \setminus \{z\})$, $d\eta = 0$ s.t.

$\int_{\alpha_i} \theta = -\int_{\alpha_i} \eta$ and apply the lemma above.

Dfn $(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ and ω area form on Σ

is s-monotone (for $s \in Spin^c(Y)$) iff $\forall x$ s.t. $s(x) = s$

$\mu : \pi_2(x,x) \to \mathbb{Z}$, $\omega : \pi_2(x,x) \to \mathbb{R}$ pos. proportional

$[\Sigma] \in \pi_2(x,x)$ has $\mu([\Sigma]) = 2$

Prop $(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ s-monotone $\Rightarrow$ strongly admissible for s

$\Rightarrow$ bound on # hom classes

s-monotonicity can be achieved in the same way as before.