# VI Cylindrical Reformulation (Lipschitz)

Take $(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ $j$ cplx structure on $\Sigma$

$J = \mathrm{Sym}^g j$;

$\mathbb{D} = \mathbb{R} \times [0,1]$,

$$\begin{tikzcd} \mathbb{D} \arrow[r, "u"] & \mathrm{Sym}^g \Sigma \\ \tilde{\mathbb{D}} \arrow[u, leftarrow, "\tilde{p}"] \arrow[r, "\tilde{u}"] & \Sigma^g \arrow[u, "\pi"] \end{tikzcd}$$

holomorphic

complete diag. by pullback

$\tilde{\mathbb{D}} \xrightarrow{\tilde{p}} \mathbb{D}$ is a branched cover of deg $g!$

$\tilde{\mathbb{D}} \xrightarrow{\tilde{u}} \Sigma^g$ holomorphic map for $\tilde{J} = j \oplus \dots \oplus j$,

$\tilde{u}$ is $S_g$-equivariant.

$\tilde{\mathbb{D}} \xrightarrow{\tilde{u}} \Sigma^g \xrightarrow{p_1} \Sigma$

proj. onto $1^{st}$ coord

$p_1 \circ \tilde{u}$ invariant by the $S_{g-1} \subseteq S_g$ fixing the $1^{st}$ coordinate in $\Sigma$

Define $\hat{\mathbb{D}} = \tilde{\mathbb{D}} / S_{g-1}$ $\quad$ $\hat{u} : \hat{\mathbb{D}} \to \Sigma$ induced maps

$\hat{p} : \hat{\mathbb{D}} \to \mathbb{D}$

so $u : \mathbb{D} \to \mathrm{Sym}^g \Sigma$ $J$-hol. corr. to a pair of maps $\hat{\mathbb{D}} \xrightarrow{\hat{p}} \mathbb{D}$ branched cover of deg $g$

$\hat{\mathbb{D}} \xrightarrow{\hat{u}} \Sigma$ holomorphic

**Prop** (O–Sz, Oh) If $\phi \in \pi_2(x,y)$ s.t.

$\mathcal{D}(\phi) = \sum a_i D_i$ has $a_i = \{0,1\}$, then $\mathcal{M}(\phi)$ can be made transversely cut out for an arbitrary a.c.s. on $\mathrm{Sym}^g \Sigma$ by an arbitrarily small generic perturbation of the $\alpha$ & $\beta$ -curves.

(perturbation of the Lagr.)

$\hat{p} : \hat{\mathbb{D}} \longrightarrow \mathbb{D}$, $\hat{u} : \hat{\mathbb{D}} \longrightarrow \Sigma \rightsquigarrow u : \mathbb{D} \to \mathrm{Sym}^g \Sigma$

$x \in \mathbb{D}$, $\{y_1, \dots, y_g\} = \hat{p}^{-1}(x)$, $u(x) = [(\hat{u}(y_1), \dots, \hat{u}(y_g))]$

what happens if we have a path $J_t = \mathrm{Sym}^g j_t$ ?

**Remark** (we need strong admissibility of the diagram)

if the path $j_t$ is close enough to constant, then $\forall \phi$ with $\mu(\phi) = 1$ intersect the diagonal in finitely many points.

① if $j_t$ constant: standard complex analysis

② having a finite nr of intersections is open in $C^1$ $C^\infty_{loc}$

③ Gromov compactness: sequence $j_t^n \to j$ for $n \to \infty$

$u_n \in \mathcal{M}_{J_t}^n(\phi)$ $\quad u_n \to u_\infty$ $\quad \mathrm{Sym}^g(j)$-hol broken strip

Now redo the same argument:

$$u : \mathbb{D} \to \mathrm{Sym}^g \Sigma \rightsquigarrow \hat{p} : \hat{\mathbb{D}} \to \mathbb{D}$$
$$\hat{u} : \hat{\mathbb{D}} \to \Sigma$$

$(\hat{p}, \hat{u}) : \hat{\mathbb{D}} \to \mathbb{D} \times \Sigma$ is $\tilde{J}$-holomorphic

where $\tilde{J}$ on $T_{(s,t,x)}(\mathbb{D} \times \Sigma) \cong \mathbb{C} \oplus T_x \Sigma$ is $i \oplus j_t$

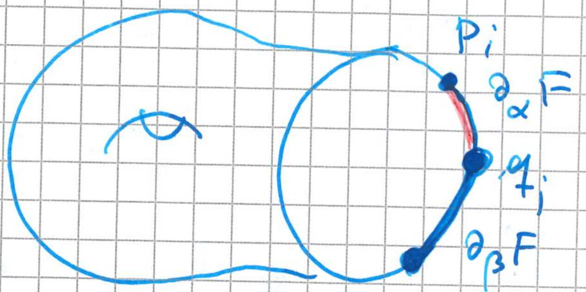
$F$ compact surface w/ bdy, oriented, no closed connected components.

$p_1, \dots, p_g$, $q_1, \dots, q_g$ marked points on $\partial F$ s.t.

- every c.c. of $\partial F$ has at least one pt of each type
- $p$'s & $q$'s alternate along $\partial F$.

$\partial_\alpha F$ portion of $\partial F$ going from $p_i$'s to $q_j$'s

$\partial_\beta F$ —— " —— $q_j$'s to $p_i$'s




$x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ $\quad x = (x_1, \dots, x_g)$, $\; y = (y_1, \dots, y_g)$

$\dot{F} = F \setminus (p \cup q)$

$B^c(x, y) =$

$\{ (F, u) \mid F$ as above, $u : \dot{F} \to \mathbb{D} \times \Sigma$ s.t.

- $u(\partial_\alpha \dot{F}) \subseteq \mathbb{R} \times \{0\} \times \underline{\alpha}$
- $u(\partial_\beta \dot{F}) \subseteq \mathbb{R} \times \{1\} \times \underline{\beta}$
- $\lim_{w \to p_i} (\pi_\Sigma \circ u)(w) = x_i$
- $\lim_{w \to p_i} (\pi_{\mathbb{R}} \circ u)(w) = -\infty$
- $\lim_{w \to q_i} (\pi_\Sigma \circ u)(w) = y_i$
- $\lim_{w \to q_i} (\pi_{\mathbb{R}} \circ u)(w) = +\infty$ $\}$

$u \in \mathcal{B}^c(x,y) \rightsquigarrow \mathcal{D}(u)$

$\mathcal{D}(u) = \sum a_i D_i$

$a_i = \# u^{-1}(\mathbb{D} \times \{z_i\})$

$u \sim v$ if $\mathcal{D}(u) = \mathcal{D}(v)$

$\pi_2^c(x,y) = \mathcal{B}^c(x,y)/\sim$

Given $\phi \in \pi_2^c(x,y)$, $\mathcal{M}^c(\phi)$ is

$\{(u,F,i) \mid (u,F) \in \phi,\ i$ cplx str. on $F$

$du + \tilde{J} \circ du \circ i = 0\} / \text{reparam}$

$(u,F,i) \sim (u',F',i')$ if

$\exists\ \varphi : F' \to F$ s.t. $u' = u \circ \varphi$, $i' = \varphi_* i$

Cylindrical versions of HF are obtained by replacing $\mathcal{M}(\phi)$ with $\mathcal{M}^c(\phi)$

---

$u \in \mathcal{M}^c(\phi) \rightsquigarrow \mathrm{ind}(u),\ \mu(\phi)$

count of sing.

$\mu(1$ If $u \in \mathcal{M}^c(\phi)$ then $\mathrm{ind}(u) + 2\delta(u) = \mu(\phi)$

## Standard Floer theory:

① $HF^\circ(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ is indep. of $j_t$ on $\Sigma$

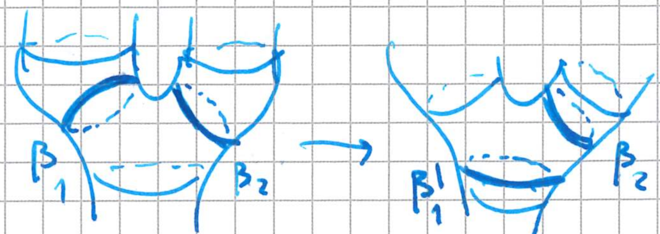
② $HF^\circ(\Sigma, \underline{\alpha}, \underline{\beta}, z) = HF^\circ(\Sigma, \underline{\alpha}', \underline{\beta}', z)$

if $\alpha' \sim \alpha$, $\beta' \sim \beta$ Ham iso. in the complement of $z$.

If $(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ and $(\Sigma', \underline{\alpha}', \underline{\beta}', z')$ are H.d. for $Y$, then they are related by a sequence of

① isotopy of the $\alpha$ & $\beta$ curves

② handleslides




③ Stabilisation


① & ② can be performed in the complement of the basepoint. If $(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ and $(\Sigma', \underline{\alpha}', \underline{\beta}', z')$ are admissible, then all intermediate diagrams are admissible.

We need to prove that each moves give an iso. of $HF^{\circ}$ groups

$(\circ - S_3)$

We also need

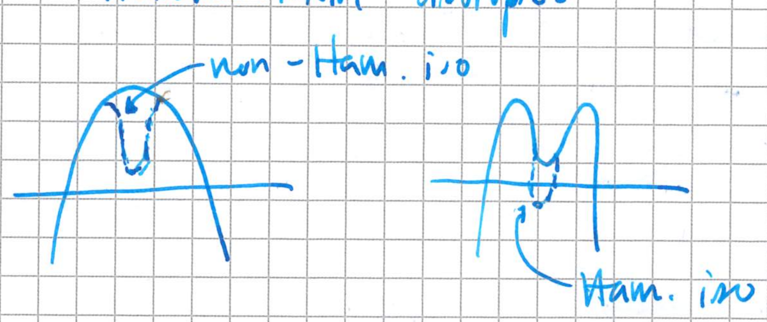
We need $\Phi_0 = \Phi_1$

"Naturality"

(Juhász, Thurston, Zemke)

$$\begin{tikzcd} (\Sigma, \underline{\alpha}, \underline{\beta}, z) \arrow[r, bend left, squiggly, "\Phi_0"] \arrow[r, bend right, squiggly, "\Phi_1"'] & (\Sigma', \underline{\alpha}', \underline{\beta}', z') \end{tikzcd}$$

① Any isotopy can be decomposed as a concatenation of isotopies where nothing happens + Hamiltonian isotopies

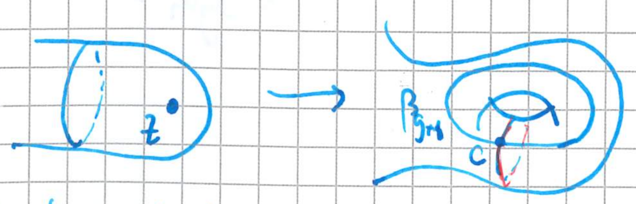



Isotopies where nothing happens are induced by diffeom. of $\Sigma$.

– trade change of $\alpha$ and $\beta$ to a change of a.c.s. $j_t$.

Ham. isotopies can be treated as usual in Floer homology.

③ Stabilisation




$(\Sigma, \alpha, \beta, z) \rightsquigarrow (\Sigma', \alpha', \beta', z) \quad \Sigma' = \Sigma \# E$

$$\mathbb{T}_\alpha \cap \mathbb{T}_\beta \overset{1:1}{\longleftrightarrow} \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$$

$$(x_1, \dots, x_g) \longleftrightarrow (x_1, \dots, x_g, c)$$

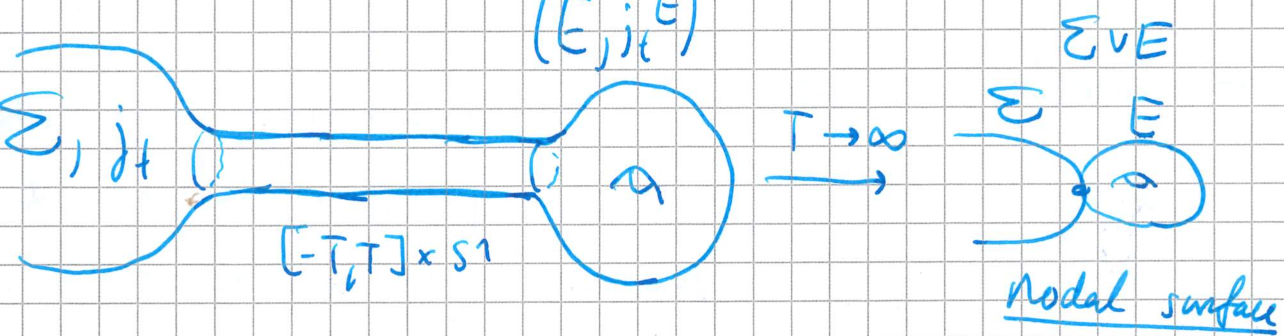
$$\pi_2(x, y) = \pi_2(x', y')$$

$$\phi \in \pi_2(x,y) \qquad \phi' \in \pi_2(x',y') \qquad n_z(\phi) = n_z(\phi')$$

$$\mu(\phi) = \mu(\phi')$$

<u>We want</u> $\#\mathcal{M}(\phi) = \#\mathcal{M}(\phi')$ if $\mu(\phi) = \mu(\phi') = 1$

Choose $j_t$ on $\Sigma$, $j_t^E$ on $E \rightsquigarrow j_t(T)$, $T \gg 0$, on $\Sigma'$




Call $\mathcal{M}_T(\phi)$ moduli spaces for $J_t(T)$

as $T \to \infty$ $\quad \mathrm{Sym}^{g+1}(\Sigma) \to \mathrm{Sym}^{g+1}(\Sigma \vee E) =$

$$= \bigcup_{k=0}^{g+1} \mathrm{Sym}^k \Sigma \times \mathrm{Sym}^{g-k+1} E \supseteq \mathbb{T}_\alpha \times \alpha_{g+1}$$

$$\mathbb{T}_\beta \times \beta_{g+1}$$

in component $\mathrm{Sym}^g(\Sigma \times E)$

$T_n \to +\infty$, $u_n \in \widehat{\mathcal{M}}_{T_n}(\phi')$ then

Gromov:

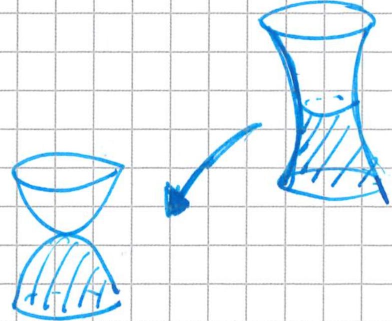
$u_n \to u_\infty$, $u_\infty$ is a strip in $\mathrm{Sym}^g \Sigma \times E$

+ spheres in the other components

strip is $u_\infty^{(0)} \times \{c\}$, where $u_\infty^{(0)}$ in $\mathrm{Sym}^g \Sigma$ in $\phi$

$u_\infty^{(0)}$ meets $V_\sigma$ in $n_z(\phi)$ many points $> 0$

There must be spheres



$(x_1, \dots, x_{g-1}, \sigma, c)$

For every int. point between $u_\infty^{(0)}$ & $V_z$

there is a holomorphic sphere in $\mathrm{Sym}^{g-1} \times \mathrm{Sym}^2 E$

passing through the same pt $(x_1, \dots, x_{g-1}, \sigma, c)$: must be

of the form $(x_1, \dots, x_g) \times S$ $S$ sphere in $\mathrm{Sym}^2 E$

Need to show two things:

1.) For every point in $Sym^2 E$ there is a unique ~~sphere~~ hol sphere

2.) Gluing theorem
(not dealt with here)

<u>for 1.)</u>:

<u>Lemma</u> $Sym^2 E$ is a $S^2$-bundle over $T^2$

$E \cong \mathbb{C}/\Lambda$ $E$ is an Abelian group

$$Sym^2 E \xrightarrow{m} E$$

$$[(x,y)] \mapsto x+y$$

$$m^{-1}(a) = [(w, a-w)]$$

$i_a : E \to E$
$w \mapsto a-w$

$m^{-1}(a) \cong E/_{i_a} \cong S^2$

Same argument: Künneth formula for $\widehat{HF}$ of connected sum of 3-mfds.