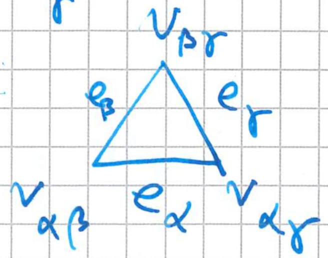
LVII

Heegaard triple diagram $(\Sigma, \underline{\alpha}, \underline{\beta}, \underline{\gamma}, z)$

$\underline{\alpha}, \underline{\beta}, \underline{\gamma}$ specify handlebodies $U_\alpha, U_\beta, U_\gamma$ on which they bound discs

Δ = triangle w. edges $e_\alpha, e_\beta, e_\gamma$ (clockwise)

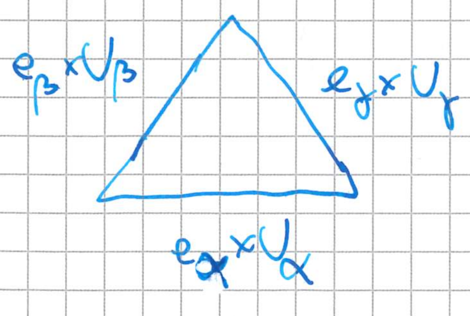


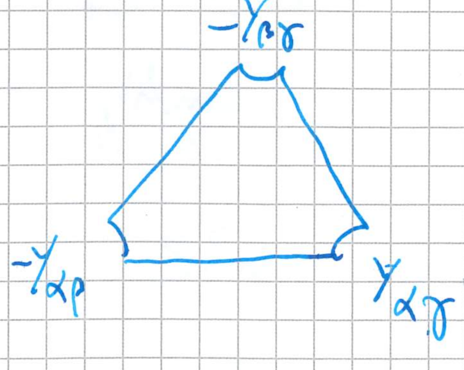
From $(\Sigma, \underline{\alpha}, \underline{\beta}, \underline{\gamma}, z) \rightsquigarrow X_{\alpha\beta\gamma}$ 4-mfd

$$X = (\Delta \times \Sigma) \cup (e_\alpha \times U_\alpha) \cup (e_\beta \times U_\beta) \cup (e_\gamma \times U_\gamma) \big/ \sim$$

we glue $\underset{\underset{\Delta \times \Sigma}{\cap}}{e_\mu \times \Sigma}$ to $e_\mu \times \partial U_\mu \subseteq e_\mu \times U_\mu$

for $\mu \in \{\alpha, \beta, \gamma\}$





$\partial X_{\alpha\beta\gamma} = -Y_{\alpha\beta} \sqcup -Y_{\beta\gamma} \sqcup Y_{\alpha\gamma}$ (pair of pants cobordism)

Use exact sequence for $(X_{\alpha\beta\gamma}, \Delta \times \Sigma)$ we obtain

$$0 \to H_2(X_{\alpha\beta\gamma}) \to H_1(\underline{\alpha}) \oplus H_1(\underline{\beta}) \oplus H_1(\underline{\gamma}) \to H_1(\Sigma) \to H_1(X_{\alpha\beta\gamma}) \to 0$$

<u>Triple periodic domain</u> a linear comb. of regions $\Sigma \setminus (\underline{\alpha} \cup \underline{\beta} \cup \underline{\gamma}) = D_0 \sqcup \ldots \sqcup D_m$

$P = \sum_{i=1}^{m} a_i D_i$ s.t. ∂P lin. comb. of α, β, γ-curves

$\Pi_{\alpha\beta\gamma}$ = group of periodic triple domains $\overset{e.s.}{\Longrightarrow}$

$$\Pi_{\alpha\beta\gamma} \cong H_2(X_{\alpha\beta\gamma})$$

$P \rightsquigarrow \hat{P} = P \cup$ discs in U_μ

bounding curves in $\underline{\mu}$, $\mu \in \{\alpha, \beta, \gamma\}$

$\mathbb{T}_\alpha, \mathbb{T}_\beta, \mathbb{T}_\gamma \subseteq \mathrm{Sym}^g \Sigma$ Heegaard tori

<u>Fix</u> $x^{\alpha\beta}, x^{\beta\gamma}, x^{\alpha\gamma}$ s.t. $x^{\mu\nu} \in \mathbb{T}_\mu \cap \mathbb{T}_\nu$

set of Whitney triangles $\mathcal{B}(x^{\alpha\beta}, x^{\beta\gamma}, x^{\alpha\gamma}) =$

$$\left\{ u : \Delta \longrightarrow \mathrm{Sym}^g \Sigma \;\middle|\; \begin{array}{ll} u(e_\mu) \subset \mathbb{T}_\mu & \mu \in \{\alpha, \beta, \gamma\} \\ u(v_{\mu\nu}) \in \mathbb{T}_\mu \cap \mathbb{T}_\nu \end{array} \right\}$$

$$u \in \mathcal{B}(x^{\alpha\beta}, x^{\beta\gamma}, x^{\alpha\gamma}) \rightsquigarrow m_{z_i}(u) = \# u^{-1}(V_{z_i})$$

$z_i \in D_i$ $\quad \mathcal{D}(u) = \sum_{i=0}^{m} m_{z_i}(u) D_i$ (shadow of u)

$$\pi_2(x^{\alpha\beta}, x^{\beta\gamma}, x^{\alpha\gamma}) = \mathcal{B}(x^{\alpha\beta}, x^{\beta\gamma}, x^{\alpha\gamma}) / \sim$$

$$u \sim v \iff \mathcal{D}(u) = \mathcal{D}(v)$$

$$x^{\alpha\beta}, x^{\beta\gamma}, x^{\alpha\gamma} \rightsquigarrow \varepsilon(x^{\alpha\beta}, x^{\beta\gamma}, x^{\alpha\gamma})$$

- fix paths on the α-curves from the components of $x^{\alpha\beta}$ to the component of $x^{\alpha\gamma}$
- on γ-curves from $x^{\alpha\gamma}$ to $x^{\beta\gamma}$
- on β-curves from $x^{\beta\gamma}$ to $x^{\alpha\beta}$

$\varepsilon(x^{\alpha\beta}, x^{\beta\gamma}, x^{\alpha\gamma})$ is a cycle on Σ

$\rightsquigarrow$ well-defined class in $H_1(X_{\alpha\beta\gamma})$

$\pi_2(x^{\alpha\beta}, x^{\beta\gamma}, x^{\alpha\gamma}) \neq \emptyset$ iff $\varepsilon(x^{\alpha\beta}, x^{\beta\gamma}, x^{\alpha\gamma}) = 0$.

$\psi \in \pi_2(x^{\alpha\beta}, x^{\beta\gamma}, x^{\alpha\gamma})$, $\psi' \in \pi_2(y^{\alpha\beta}, y^{\beta\gamma}, y^{\alpha\gamma})$

$$\partial(\mathcal{D}(\psi) - \mathcal{D}(\psi')) = \varepsilon(x^{\alpha\beta}, x^{\beta\gamma}, x^{\alpha\gamma}) - \varepsilon(y^{\alpha\beta}, y^{\beta\gamma}, y^{\alpha\gamma})$$

$$= \overbrace{\varepsilon(y^{\alpha\beta}, x^{\alpha\beta})}^{\in H_1(Y_{\alpha\beta})} + \overbrace{\varepsilon(y^{\beta\gamma}, x^{\beta\gamma})}^{\in H_1(Y_{\beta\gamma})} + \underbrace{\varepsilon(x^{\alpha\gamma}, y^{\alpha\gamma})}_{\in H_1(Y_{\alpha\gamma})}$$

+ multiples of $\alpha_i, \beta_i, \gamma_i$

Then $\mathcal{D}(\psi') - \mathcal{D}(\psi)$ defines a class $[\psi' - \psi]$ in $H_2(X_{\alpha\beta\gamma}, \partial X_{\alpha\beta\gamma})$

Two Whitney triangles ψ', ψ are <u>Spin^c-eq.</u>

if $\mathcal{D}(\psi') - \mathcal{D}(\psi) = \mathcal{D}(\phi_{\alpha\beta}) + \mathcal{D}(\phi_{\beta\gamma}) + \mathcal{D}(\phi_{\alpha\gamma})$

$\phi_{\mu\nu}$: Whitney strips in $Y_{\mu\nu}$

$\Leftrightarrow$ $[\psi' - \psi] = 0$

we can associate $\psi \longmapsto s(\psi) \in \mathrm{Spin}^c(X_{\alpha\beta\gamma})$

$\mathrm{Spin}^c(X_{\alpha\beta\gamma}) = \{$ a.c.s. on 3-skel of $X_{\alpha\beta\gamma}\}$ / homotopy on the two-skeleton

$\big(s(\psi') - s(\psi) = \pm \mathrm{Pd}([\psi' - \psi])$

$$\left.\mathrm{Spin}^c(X_{\alpha\beta\gamma}) \xrightarrow{c_1} H^2(X_{\alpha\beta\gamma})\right)_{\mu\in\{\alpha,\beta,\gamma\}}$$

Fix $u \in \Psi$ generic, fix $f_\mu : U_\mu \to \mathbb{R}$ Morse w. unique min / max on Σ

$F_0, F_1 \subseteq X_{\alpha\beta\gamma}$: $F_0 \cap \Delta \times \Sigma = \Delta \times \{z\}$

$F_0 \cap e_\mu \times U_\mu = e_\mu \times \gamma_z^\mu$

γ_z^μ = unique traj of f_μ from min to z.

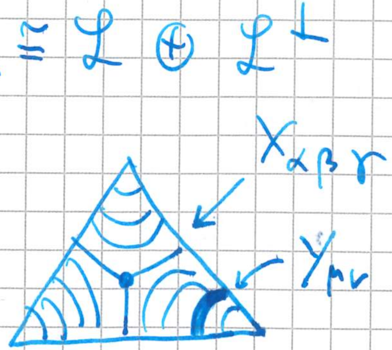
$$F_1 \cap \Delta \times \Sigma = \{(z, u(z)) ; z \in \Delta\}$$

$$F_1 \cap e_\mu \times U_\mu = \bigcup_{z\in e_\mu} \{\{z\} \times \Gamma^\mu(u(z))\}$$

$\uparrow$ g-tuple of gradient flow traj.: go from ind 1 crit pts to g pts of $u(z)$

On $X_{\alpha\beta\gamma} \setminus (F_0 \cup F_1)$ $TX_{\alpha\beta\gamma} \cong \mathcal{L} \oplus \mathcal{L}^\perp$

$$\mathcal{L} = \begin{cases} T\Sigma & \text{on } \Delta\times\Sigma \\ \nabla f_\mu^\perp & \text{on } e_\mu \times U_\mu \end{cases}$$



$Y_{\mu\nu}(t)$ $(t\in(0,1))$ the preimage.

$\partial(F_0 \vee F_1)$ $(g+1)$-tuple of paths for all t but a finite number.

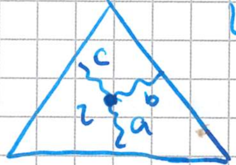
$\psi \in \pi_2(x^{\alpha\beta}, x^{\beta\gamma}, x^{\alpha\gamma})$ $\quad P \subseteq \Pi_{\alpha\beta\gamma}$ triple periodic domain

$e(P)$ Euler measure: $P = \sum a_i D_i$

$$e(P) = \sum a_i \, e(D_i)$$

$$e(D_i) = \chi(D) - \frac{\#\text{vertices}(D_i)}{4}$$

$\sigma(\psi, P)$ "spider number"



$u(z) = \{p_1, \dots, p_g\}$ $\quad u(a) = \{a_1, \dots, a_g\}$

$u(b) = \{b_1, \dots, b_g\}$

$u(c) = \{c_1, \dots, c_g\}$

$$\underline{n}_q(P) = n_{q_1}(P) + \dots + n_{q_g}(P)$$

$$\partial P = \partial_\alpha P + \partial_\beta P + \partial_\gamma P$$

$\partial_\alpha P \cdot \underline{a}$ = sum of the alg. intersection between $\partial_\alpha P$ and a_i where intersection at the endpoint of a_i counts half.

$\sigma(\psi, P) = \underline{n}_q(P) + \partial_\alpha P \cdot \underline{a} + \partial_\beta P \cdot \underline{b} + \partial_\gamma P \cdot \underline{c}$

Finally $\langle c_1(s(\psi)), \hat{P}\rangle = e(P) + 2\sigma(\psi, P)$.

If $\psi \in \pi_2(x^{\alpha\beta}, x^{\beta\gamma}, x^{\alpha\gamma})$, $P \in \Pi_{\alpha\beta}$ then

R.H.S. $= e(P) + 2\underline{n}_{x^{\alpha\beta}}(P)$

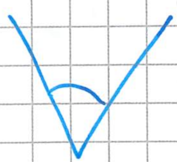
$\langle c_1(\psi'), \hat{P}\rangle - \langle c_1(\psi), \hat{P}\rangle = 2[\psi' - \psi]\cdot \hat{P}$

$\psi \in \pi_2(x^{\alpha\beta}, x^{\alpha\gamma}, x^{\beta\gamma})$, $\mathcal{M}(\psi) = \{u \in \mathcal{B}(x^{\alpha\beta}, x^{\beta\gamma}, x^{\alpha\gamma}) \mid$

$du + J_z(u) \circ du \circ i = 0\}$

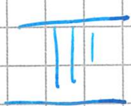
$z \in \Delta$

$z \mapsto j_z$ cplx str. on Σ, $J_z = \mathrm{Sym}^g j_z$



$\alpha\beta$

$\beta\gamma$



$\alpha\gamma$

$\mu : \pi_2(x^{\alpha\beta}, x^{\beta\gamma}, x^{\alpha\gamma}) \to \mathbb{Z}$ Maslov index

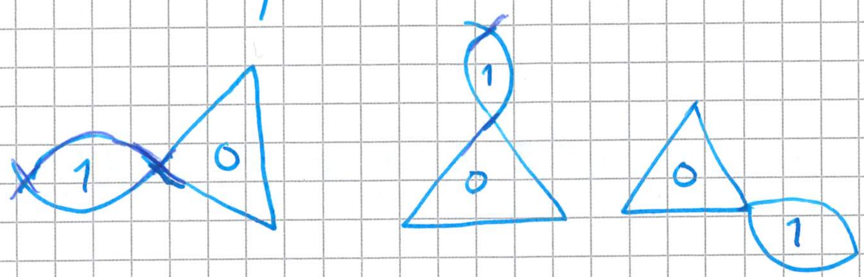
(= Fredholm index of lin CR-op.)

(there's a combinatorial formula due to Sarkar)

<u>Thm</u> For a generic path j_z on Σ we have: • if $\mu(\psi)<0$ then $\mathcal{M}(\psi)=\emptyset$

• If $\mu(\psi)=0$, then $\mathcal{M}(\psi)$ is a finite set

• If $\mu(\psi)=1$, then $\mathcal{M}(\psi)$ is a one-dim mfd which can be compactified



Given $s\in Spin^c(X_{\alpha\beta\gamma})$ $x^{\alpha\beta}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$

$x^{\beta\gamma}\in\mathbb{T}_\beta\cap\mathbb{T}_\gamma$ s.t.

$s(x^{\alpha\beta})=s^{\alpha\beta}=s|_{Y_{\alpha\beta}}$, $s(x^{\beta\gamma})=s^{\beta\gamma}=s|_{Y_{\beta\gamma}}$

$$f^\infty_{\alpha\beta\gamma,s}: CF^\infty_{s_{\alpha\beta}}(\Sigma,\underline{\alpha},\underline{\beta},\underline{\gamma},z)\otimes CF^{\leq 0}_{s_{\beta\gamma}}(\Sigma,\underline{\beta},\underline{\gamma},z)$$

$$\longrightarrow CF^\infty_{s_{\alpha\gamma}}(\Sigma,\underline{\alpha},\underline{\gamma},z)$$

CF^∞ gen by $[x,i]$, $C^{\leq 0}$ gen by $i\leq 0$

similarly for f^+

$$\hat{f}_{\alpha\beta\gamma,s} : \widehat{CF}_{s_{\alpha\beta}}(\Sigma, \underline{\alpha}, \beta, z) \otimes \widehat{CF}_{s_{\beta\gamma}}(\Sigma, \beta, \gamma, z)$$

$$\to \widehat{CF}_{s_{\alpha\gamma}}(\Sigma, \underline{\alpha}, \gamma, z)$$

$$f^{\infty}_{\alpha,\beta,\gamma,s}\left([x^{\alpha\beta}, i] \otimes [x^{\beta\gamma}, j]\right) =$$

$$= \sum_{\substack{x^{\alpha\gamma} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma \\ s(x^{\alpha\gamma}) = s^{\alpha\gamma}}} \sum_{\substack{\psi \in \pi_2(x^{\alpha\beta}, x^{\beta\gamma}, x^{\alpha\gamma}) \\ \mu(\psi) = 0 \\ s(\psi) = s}} \# \mathcal{M}(\psi) [x^{\alpha\gamma}, i+j-n_z(\psi)]$$

To make sense of this one needs admissibility

weak admissibility $\Leftarrow$ $\mathbb{T}_\alpha$ $\mathbb{T}_\beta$ $\mathbb{T}_\gamma$ exact in $\mathrm{Sym}^g \Sigma \setminus V_z$

strong admissibility $\Leftarrow$ $\mathbb{T}_\mu, \mathbb{T}_\nu$

$\{\mu, \nu\} \subseteq \{\alpha, \beta, \gamma\}$ $s^{\mu\nu}$-monotone

i.e. α, β, γ exact in $\Sigma \setminus \{z\}$

If we have strong adm. for $\underline{s}$ $\Rightarrow$ then $f^{\circ}_{\alpha\beta\gamma}$ is a chain map

$(\circ = \pm, \infty)$, weak admissibility $\Rightarrow$ $\hat{f}$ is a chain map