L VIII

quadruple Heegaard diagram:

$(\Sigma, \underline{\alpha}, \underline{\beta}, \underline{\gamma}, \underline{\delta}\ z) \rightsquigarrow X_{\alpha\beta\gamma\delta}$ 4-mfd

$$\partial X_{\alpha\beta\gamma\delta} = -Y_{\alpha\beta} \sqcup -Y_{\beta\gamma} \sqcup -Y_{\gamma\delta} \sqcup Y_{\alpha\delta}$$

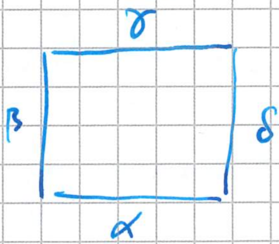
<u>Notation:</u> χ rect.
ψ triang.
ϕ strip

$\square \times \Sigma$

$\mathcal{B}(x^{\alpha\beta}, x^{\beta\gamma}, x^{\gamma\delta}, x^{\alpha\delta})$ $\quad x^{\mu\nu} \in \mathbb{T}_\mu \cap \mathbb{T}_\nu$

smooth maps $\square \to \mathrm{Sym}^g \Sigma$ bdy on $\mathbb{T}_\alpha, \mathbb{T}_\beta, \mathbb{T}_\gamma, \mathbb{T}_\delta$

and vertices to $x^{\mu\nu}$



$u \in \mathcal{B}(-) \rightsquigarrow \mathcal{D}(u)$ shadow

$\pi_2(x^{\alpha\beta}, x^{\beta\gamma}, x^{\gamma\delta}, x^{\alpha\delta}) = \mathcal{B}(-)/\sim$ $\qquad u \sim v \iff \mathcal{D}(u) = \mathcal{D}(v)$

$\mathcal{D}(\chi_0) - \mathcal{D}(\chi_1)$ defines a class in $H_2(X_{\alpha\beta\gamma\delta}, \partial X_{\alpha\beta\gamma\delta})$

denoted by $[\chi_0 - \chi_1]$

<u>Dfn</u> χ_0, χ_1 "Spin^c-eq." if $[\chi_0 - \chi_1] = 0$

i.e $\mathcal{D}(\chi_0) = \mathcal{D}(\chi_1) + \mathcal{D}(\phi_1) + \mathcal{D}(\phi_2) + \mathcal{D}(\phi_3) + \mathcal{D}(\phi_4)$

$\phi_1 \in \pi_2(x_{\alpha\beta}, y_{\alpha\beta})$ and so on

$\psi_0 \in \pi_2(x^{\alpha\beta}, x^{\beta\gamma}, x^{\alpha\gamma})$ $\quad \psi_1 \in \pi_2(x^{\alpha\gamma}, x^{\gamma\delta}, x^{\alpha\delta})$

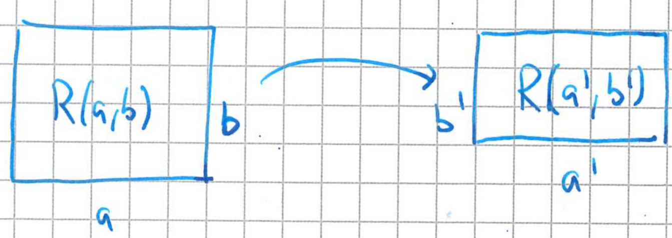
we "concatenate" them $\quad \chi = \psi_0 * \psi_1 \quad D(\chi) = D(\psi_0) + D(\psi_1)$

$\psi_0' \in \pi_2(y^{\alpha\beta}, y^{\beta\gamma}, y^{\alpha\gamma})$, $\quad \psi_1' \in \pi_2(y^{\alpha\gamma}, y^{\gamma\delta}, y^{\alpha\delta})$

If $\quad [\psi_0 - \psi_0'] = [\psi_1 - \psi_1'] = 0 \quad$ then

$[\psi_1 * \psi_0 - \psi_1' * \psi_0'] \in \mathrm{Im}\left(H_2(Y_{\alpha\gamma}) \to H_2(X_{\alpha\beta\gamma\delta}, \partial X_{\alpha\beta\gamma\delta})\right)$

Rectangles have a 1-parameter (real) family of conformal structures



$\exists$ biholomorphism $R(a,b) \to R(a',b') \iff \frac{a}{b} = \frac{a'}{b'}$

($\Rightarrow$ by Schwarz reflection & classifications of $\mathbb{C} \xrightarrow{\cong} \mathbb{C}$)

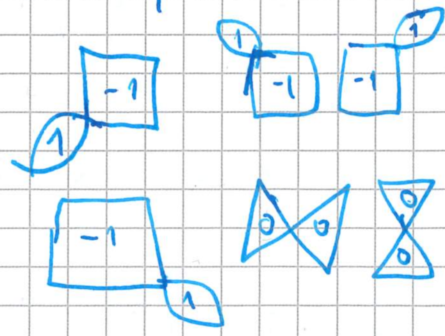
Given χ homology class of rectangles

$\mathcal{M}(\chi) = \{ u \in \chi$ which are holomorphic for <u>some</u> conformal structure on the rectangle. $\}$

For a generic family J_z, $z \in \square$ of cpx structures on Σ, we have

- $\mu(\underset{\chi}{\ }) = \emptyset$ if $\mu(\underset{\chi}{\ }) < -1$
- finite set if $\mu(\chi) = -1$
- 1-dim. mfd if $\mu(\chi) = 0$

Case $\mu(\chi) = 0$:



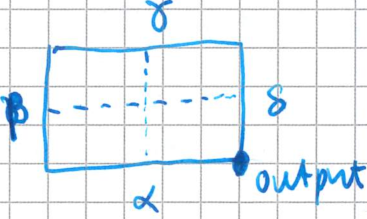
We define $\hat{H} : \widehat{CF}(\Sigma, \underline{\alpha}, \underline{\beta}, z) \otimes \widehat{CF}(\Sigma, \underline{\beta}, \underline{\gamma}, z)$

$\otimes\, \widehat{CF}(\Sigma, \underline{\gamma}, \underline{\delta}, z) \to \widehat{CF}(\Sigma, \underline{\alpha}, \underline{\delta}, z)$

$$\hat{H}(x^{\alpha\beta} \otimes x^{\alpha\gamma} \otimes x^{\gamma\delta}) = \sum_{x^{\alpha\delta} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta} \ \sum_{\substack{\chi \in \pi_2(\ ) \\ \mu(\chi) = -1 \\ n_z(\chi) = 0}} \# \mathcal{M}(\chi)\, x^{\alpha\delta}$$

<u>Prop</u> $\hat{H}$ is a homotopy between $f_{\alpha\gamma\delta}(f_{\alpha\beta\gamma} \otimes id)$ and $f_{\alpha\beta\delta}(id \otimes f_{\beta\gamma\delta})$

i.e. $\partial H(x^{\alpha\beta} \otimes x^{\beta\gamma} \otimes x^{\gamma\delta})$



$+ H(\partial x^{\alpha\beta} \otimes x^{\beta\gamma} \otimes x^{\gamma\delta})$

$+ H(x^{\alpha\beta} \otimes \partial x^{\beta\gamma} \otimes x^{\gamma\delta}) + H(x^{\alpha\beta} \otimes x^{\beta\gamma} \otimes \partial x^{\gamma\delta})$

$= f_{\alpha\gamma\delta}(f_{\alpha\beta\gamma} \otimes id) + f_{\alpha\beta\delta}(id \otimes f_{\beta\gamma\delta})$

if $[\Psi_0 - \Psi_0'] = [\Psi_1 - \Psi_1'] = 0$ then

$$[\Psi_1 * \Psi_0 - \Psi_1' * \Psi_0'] \in \mathrm{Im}(H_2(Y_{\alpha\gamma}) \to H_2(X_{\alpha\beta\gamma\delta}, \partial X_{\alpha\beta\gamma\delta}))$$

when defining H we fix $\mathcal{S}$ a set of Spin^c-eq. classes of rectangles that differ by elements of $H_2(Y_{\alpha\gamma}) + H_2(Y_{\beta\delta})$

$s \in \mathcal{S}$, $s_{\mu\nu}$ restriction to $Y_{\mu\nu}$

<u>We want</u>:
- weak admissibility for hat version
- strong admissibility for $+, -, \infty$ versions

exactness in $\Sigma \setminus \{z\} \Rightarrow$ weak admissibility

- $s_{\mu\nu}$ - monotonicity for all $\{\mu, \nu\} \subseteq \{\alpha, \beta, \gamma, \delta\}$ $\Rightarrow$ strong admissibility.

Topological conclusion: the compositions

$$H_2(Y_{\alpha\gamma}) \to H_2(X_{\alpha\beta\gamma\delta}, \partial X_{\alpha\beta\gamma\delta}) \to H_1(Y_{\beta\delta})$$

$$H_2(Y_{\beta\delta}) \to H_2(X_{\alpha\beta\gamma\delta}, \partial X_{\alpha\beta\gamma\delta}) \to H_1(Y_{\alpha\gamma})$$

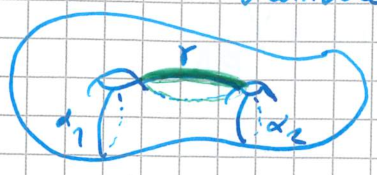
both trivial $(\Rightarrow s_{\mu\nu}$ don't depend on $s \in \mathcal{S}$

In general one defines:

$$H : CF^{\circ}(\Sigma, \underline{\alpha}, \underline{\beta}, z) \otimes CF^{\leq 0}(\Sigma, \underline{\beta}, \underline{\gamma}, z) \otimes CF^{\leq 0}(\Sigma, \underline{\gamma}, \underline{\delta}, z) \to CF^{\circ}(\Sigma, \underline{\alpha}, \underline{\delta}, z)$$

Handleslide invariance

handlebody



compressing discs

handleslide replace one of α's by γ

Heegaard diagrams

$(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ describing Y

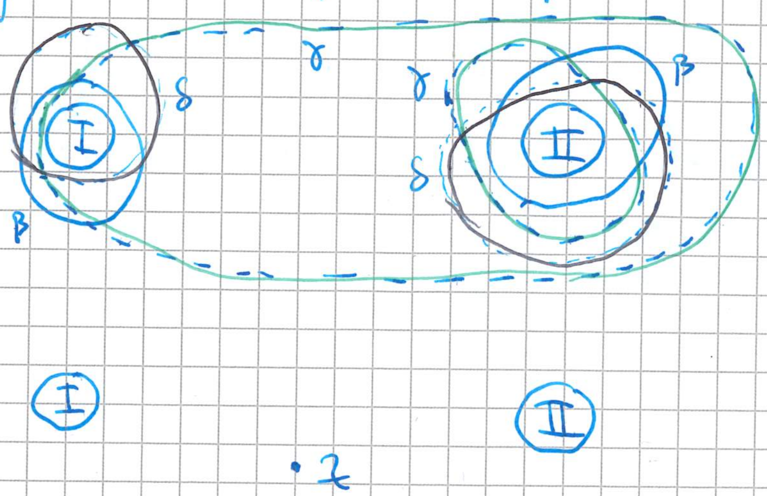
$(\Sigma, \underline{\alpha}, \underline{\gamma}, z)$ describing Y

$\gamma_1, \dots, \gamma_g$ differ from $\beta_1, \dots, \beta_g$ by a handleslide

more precisely: $\gamma_1, \dots, \gamma_{g-1}$ small Ham. deformation of

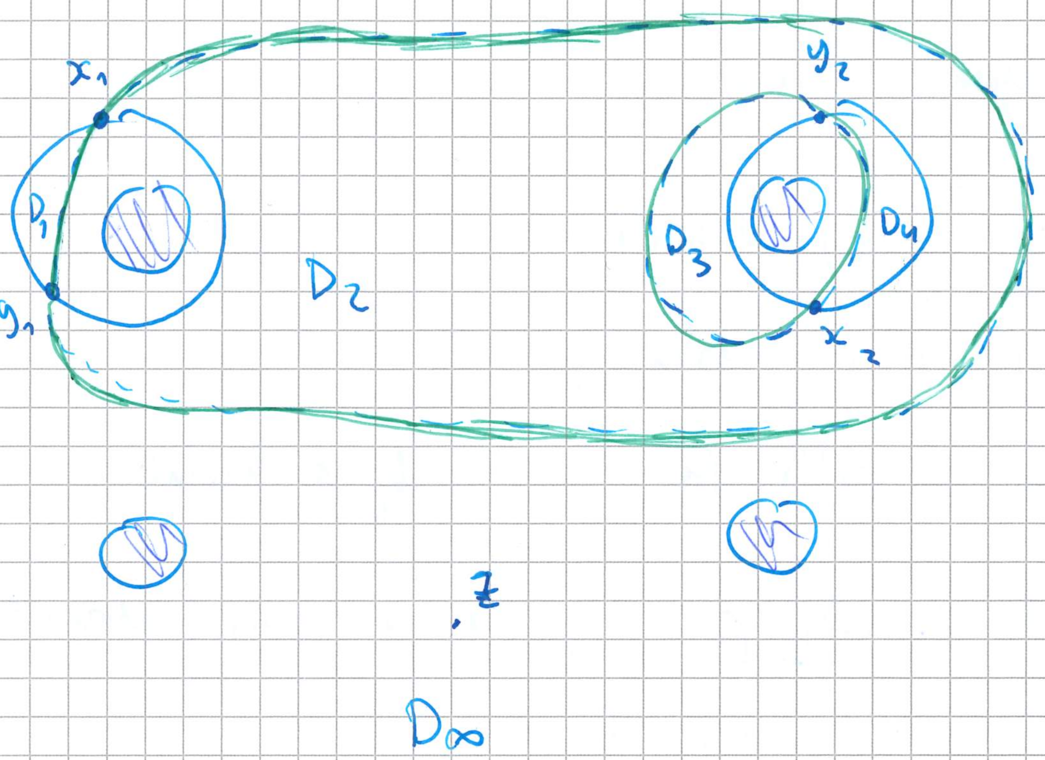
$\beta_1, \dots, \beta_{g-1}$

Fix $g(\Sigma) = 2$, β, γ, δ pairs of curves



$$Y_{\beta,\gamma} = Y_{\gamma\delta} = Y_{\beta\gamma} = (S^2 \times S^1) \# (S^2 \times S^1)$$

Look at $(\Sigma, \beta, \gamma, z)$ first



Generators: $x_{--} = (x_1, x_2)$, $x_{+-} = (y_1, x_2)$

$x_{-+} = (x_1, y_2)$, $x_{++} = (y_1, y_2)$

homotopy classes of discs $\mu(\phi) = 1$, $n_z(\phi) = 0$

$\phi_1, \phi_2, \bar{\phi}_2 \in \pi_2(x_{--}, x_{+-})$, $\phi_1', \phi_2', \bar{\phi}_2' \in \pi_2(x_{-+}, x_{++})$

$\phi_3, \phi_4 \in \pi_2(x_{--}, x_{-+})$, $\phi_3', \phi_4' \in \pi_2(x_{+-}, x_{++})$

$\mathcal{D}(\phi_1) = \mathcal{D}(\phi_1') = D_1 \quad \leftarrow$

$\mathcal{D}(\phi_2) = \mathcal{D}(\phi_2') = D_2 + D_3 \quad \leftarrow$

$\mathcal{D}(\Phi) = \mathcal{D}(\Phi_2') = D_2 + D_4 \quad \leftarrow$

$\mathcal{D}(\phi_3) = \mathcal{D}(\phi_3') = D_3$

$\mathcal{D}(\phi_4) = \mathcal{D}(\phi_4') = D_4$

Lemma $\# \hat{\mathcal{M}}(\phi_1) = 1$

$u \in \mathcal{M}(\phi) \iff$

$\hat{u} : \hat{\mathbb{D}} \to \Sigma$

$\hat{p} : \hat{\mathbb{D}} \to \mathbb{D}$ 2-fold branched cover

Proof

In this case $\hat{\mathbb{D}} = \mathbb{D} \sqcup \mathbb{D}$, $\hat{u}$ maps 1st copy to D_1 with degree 1, 2nd copy constantly to x_2.

It remains to count maps $\hat{u} : \mathbb{D} \to D_1$

$\exists$ bihol. $\hat{u} : \mathbb{D} \to D_1$

Two such maps differ by automorphism of $\mathbb{D}$ i.e. translation in the s-direction.

$\Rightarrow \hat{\mathcal{M}}(\phi_1) = 1$

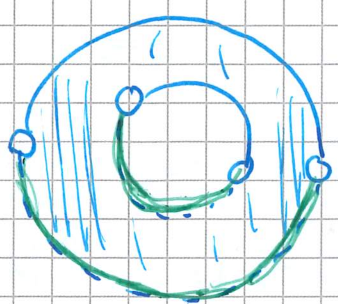
Lemma $\# \mathcal{M}(\phi_2) + \# \mathcal{M}(\bar{\Phi}_2) = 1$

$u \in \mathcal{M}(\phi_2)$ then $\hat{u} : \hat{\mathbb{D}} \to \Sigma$

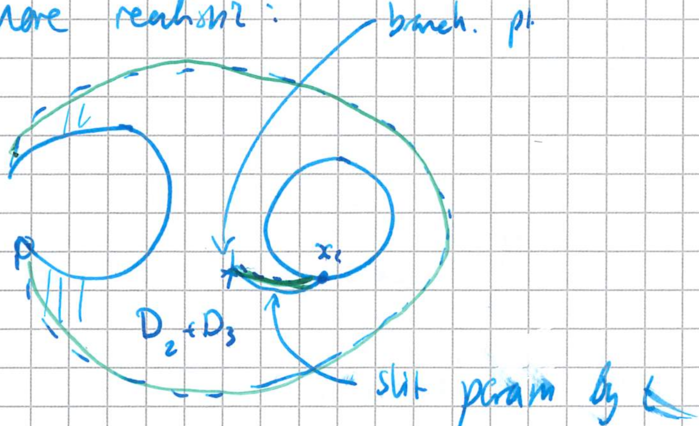
$\hat{p} : \hat{\mathbb{D}} \to \mathbb{D}$ 2 : 1 branched

$\hat{u}$ covr $D_2 + D_3$ w. deg 1 $\Rightarrow$ $\hat{\mathbb{D}}$ annulus

w. two marked pts in each bdy component.



more realistic:



We have a one-param. family $\hat{u} : \hat{\mathbb{D}} \to \mathbb{D}$ we can choose from

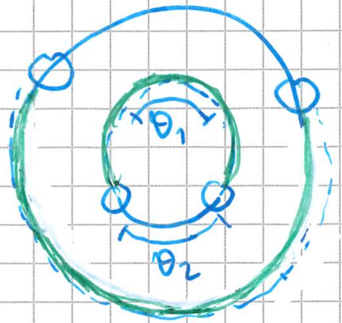
For how many of them $\exists$ $\hat{p} : \hat{\mathbb{D}} \to \mathbb{D}$?

$t \in (0,1)$, $\theta_1(t)$ = angle swept out by the arc in $\partial \hat{\mathbb{D}}$ mapped to β_1 (same for θ_2 & β_2)

parametrises the pos. of the branched pt in the slit

$\exists \; \hat{p} : \hat{\mathbb{D}} \to \mathbb{D}$ iff $\exists$ an involution of $\hat{\mathbb{D}}$

exchanges the bdy components, and maps the above to blue arc



Elementary complex analysis:

$\exists$ iff $\theta_1(t) = \theta_2(t)$

$\hat{\mathbb{D}} = \{ z \in \mathbb{C} \mid \frac{1}{r} \leq |z| \leq r \}$

$\mathrm{Aut}(\hat{\mathbb{D}}) = \left\{ \begin{array}{l} z \longmapsto cz, \quad |c| = 1 \\ z \longmapsto \frac{c}{|z|} \end{array} \right.$

For $t = 0$, $\theta_2(0) = 2\pi$, $\theta_1(0) = \nu_0 < 2\pi$

$t = 1$, $\theta_2(1) = \nu_2$, $\theta_1(1) = \nu_1$

For $t \to 1$ $(\hat{\mathbb{D}}_t, \hat{u}_t)$ converge to

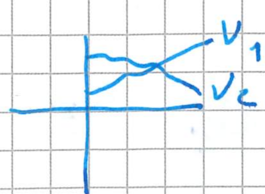
$(\hat{\mathbb{D}}_1, \hat{u}_1)$ $\hat{u}_1 : \hat{\mathbb{D}}_1 \to D_2$ bihol

by Gromov compactness

We have an odd nr. of t s.t. $\theta_1(t) = \theta_2(t)$ if

$\nu_1 > \nu_2$

even if $\nu_2 < \nu_1$



$\widehat{HF}(\Sigma, \beta, \gamma, z) = H_*(S^1)^{\otimes 2}$

$(\Sigma, \beta, \delta, z)$ also the same thing

In $CF^{\leq 0}(\Sigma, \beta, \gamma, z)$ generator $\Theta = [x_{--}, 0]$ is a cycle.

$\widehat{HF}(\Sigma, \alpha, \beta, z) \cong \widehat{HF}(\Sigma, \alpha, \gamma, z)$

$(\Sigma, \alpha, \beta, \gamma, z) \rightsquigarrow f_{\alpha\beta\gamma} : CF^{0}(\Sigma, \alpha, \beta, z) \otimes CF^{\leq 0}(\Sigma, \beta, \gamma, z)$
$\to CF^{0}(\Sigma, \alpha, \gamma, z)$

$f : CF^{0}(\Sigma, \alpha, \beta, z) \to CF^{0}(\Sigma, \alpha, \gamma, z) \to CF(\Sigma, \alpha, \delta, z)$

$f([x, i]) = f_{\alpha\beta\gamma}(x^{\alpha\beta} \otimes [\Theta, 0]^{\beta\gamma})$ chain map

$f_{\alpha\gamma\delta}(x^{\alpha\gamma} \otimes \Theta^{\gamma\delta})$

continuation

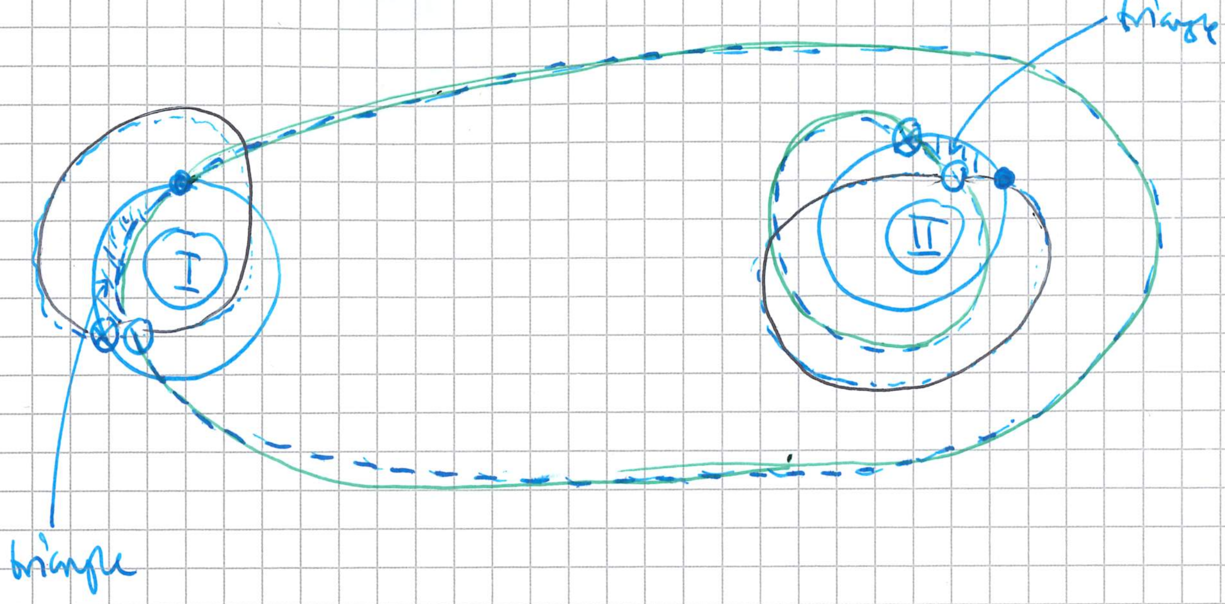
$f_{\alpha\gamma\delta}(f_{\alpha\beta\gamma}(x \otimes \Theta^{\beta\gamma}) \otimes \Theta^{\gamma\delta}) \simeq$

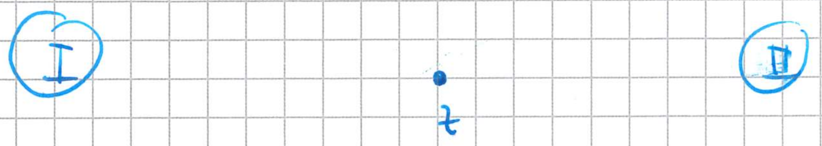
element of ham. iso.

$f_{\alpha\beta\delta}(x^{\alpha\beta} \otimes f_{\beta\gamma\delta}(\Theta^{\beta\gamma} \otimes \Theta^{\gamma\delta})) = f_{\alpha\beta\delta}(x^{\alpha\beta} \otimes \Theta^{\beta\delta})$ (blue to ~~~~)

HENCE: $f_{\alpha\beta\delta}$ is iso. in homology

$\Downarrow$


triangle
I
II
triangle


I
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