L. IX

(A, ∂_A), (B, ∂_B) chain complexes

$f : A \to B$ linear map

$D : A \oplus B \to A \oplus B$ $\qquad D = \begin{pmatrix} \partial_A & 0 \\ f & \partial_B \end{pmatrix}$

<u>Lemma:</u> $D^2 = 0 \iff f \circ \partial_A = \partial_B \circ f$ (i.e. f is a chain map)

<u>Def</u> $f : A \to B$ is a chain map

$(A \oplus B, D)$ is called the cone of f $(C(f))$

<u>Lemma</u> There is an exact sequence of complexes

$$0 \to B \xrightarrow{b \mapsto (0,b)} C(f) \xrightarrow{(a,b) \mapsto a} A \to 0$$

$\Rightarrow$ exact triangle

$$\begin{tikzcd} H(A) \arrow[rr, "f_*"] & & H(B) \arrow[dl] \\ & H(C(f)) \arrow[ul] & \end{tikzcd}$$

(A, ∂_A), (B, ∂_B), (C, ∂_C) chain complexes

$f : A \to B$, $g : B \to C$, $h : A \to C$

$$D = \begin{pmatrix} \partial_A & 0 & 0 \\ f & \partial_B & 0 \\ h & g & \partial_C \end{pmatrix}$$

Lemma $D^2 = 0$ iff
- f, g chain maps
- h is a homotopy between $g \circ f$ and 0

i.e. $\partial_C h + h \partial_A = g \circ f$

"Double cone of f & g"

Lemma If $H_*(A \oplus B \oplus C, D) = 0$, then

$A \overset{q.is.}{\simeq} C(g) \Rightarrow \exists$ exact triangle

$$\begin{tikzcd} H(A) \arrow[rr, "f_*"] & & H(B) \arrow[dl, "g_*"] \\ & H(C) \arrow[ul] & \end{tikzcd}$$

Proof $D = \begin{pmatrix} \partial_A & 0 & 0 \\ f & \partial_B & 0 \\ h & g & \partial_C \end{pmatrix}$

$B \oplus C$ is a subcomplex, i.e. $C(g)$

So $(A \oplus B \oplus C, D)$ is the cone of $(f, h) : A \to C(g)$

$$\begin{tikzcd} H(A) \arrow[rr] & & H(C(g)) \arrow[dl] \\ & H(A \oplus B \oplus C, D) \arrow[ul] & \end{tikzcd}$$

$$H(A \oplus B \oplus C, D) = 0$$

$$\begin{tikzcd}
H(B) \arrow[rr, "g_*"] & \arrow[d, phantom, "\circlearrowright"] & H(C) \arrow[dl] \arrow[ddl, bend left] \\
& H(C(g)) \cong H(B \oplus C) \arrow[ul] & \\
& H(A) \arrow[u, "\cong"', "(f\ h)_*"] \arrow[uul, bend left, "f_*"] &
\end{tikzcd}$$

<u>Dfn</u> (C, ∂) is a filtered complex if it admits a sequence of subcomplexes

$$\{0\} \subseteq C_0 \subseteq C_1 \subseteq \dots \subseteq C_m \subseteq \dots \subseteq C$$

C_i subcomplex of C

A, B filtered complexes, a <u>filtered chain map</u> is a chain map $f: A \to B$ s.t.

$$f(A_i) \subseteq B_i$$

<u>Example</u> $(A \oplus B \oplus C, D)$ is filtered

$$\{0\} \subseteq C \subseteq B \oplus C \subseteq A \oplus B \oplus C$$

<u>Lemma</u> A, B filtered complexes, $f: A \to B$ filtered chain map

$f_i: A_i/A_{i-1} \to B_i/B_{i-1}$ induced chain maps

If $(f_i)_*: H(A_i/A_{i-1}) \to H(B_i/B_{i-1})$ are isom $\forall i$

then $f_*: H(A) \to H(B)$ isom.

<u>Proof:</u>

<u>Step 1</u> $(f|_{A_i})_*: H(A_i) \to H(B_i)$ isom. for all i:

for $i=0$, $f|_{A_0} = f_0$ q.is. by hypothesis

Assume isom. up to $i=k \Rightarrow$ isom $i=k+1$

$$\begin{tikzcd}
0 \arrow[r] & A_k \arrow[r] \arrow[d, "f|_{A_k}"] & A_{k+1} \arrow[r] \arrow[d, "f|_{A_{k+1}}"] & A_{k+1}/A_k \arrow[r] \arrow[d, "f_{k+1}"] & 0 \\
0 \arrow[r] & B_k \arrow[r] & B_{k+1} \arrow[r] & B_{k+1}/B_k \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
H(A_{k+1}/A_k) \arrow[r] \arrow[d, "\cong\ (f_{k+1})_*"] & H(A_k) \arrow[r] \arrow[d, "\cong\ (f_k)_*"] & H(A_{k+1}) \arrow[r] \arrow[d, "(f_{k+1})_*"] & H(A_{k+1}/A_k) \arrow[r] \arrow[d, "\cong\ (f_{k+1})_*"] & H(A_k) \arrow[d, "(f_k)_*"] \\
H(B_{k+1}/B_k) \arrow[r] & H(B_k) \arrow[r] & H(B_{k+1}) \arrow[r] & H(B_{k+1}/B_k) \arrow[r] & H(B_k)
\end{tikzcd}$$

isom by 5-lemma

$\Rightarrow (f_{k+1})_*: H(A_{k+1}) \to H(B_{k+1})$

isom for all $k \geq 0$.

$$\begin{tikzcd}
H(\varinjlim A_i) \arrow[r, phantom, "\cong"] \arrow[d, "(\varinjlim (f|_{A_i}))_*"] & H(A) \arrow[d, "f_*"] \\
H(\varinjlim B_i) \arrow[r, phantom, "\cong"] & H(B)
\end{tikzcd}$$

$$H(\varinjlim A_i) \cong \varinjlim H(A_i)$$

$$\text{— " —} \quad B_i \qquad \text{— " —} \qquad B_i$$

$$\Rightarrow \quad \begin{tikzcd}
\varinjlim H(A_i) \arrow[r, "\cong"] \arrow[d, "\varinjlim (f|_{A_i})_*"] & H(A) \arrow[d, "f_*"] \\
\varinjlim H(B_i) \arrow[r, "\cong"] & H(B)
\end{tikzcd} \quad \Rightarrow \text{isom}$$

$\square$

Lemma (Seidel's cone detection lemma)

$$\begin{tikzcd}
A_0 \arrow[r, "f_0"] \arrow[rr, bend right, "h_0"'] & A_1 \arrow[r, "f_1"] \arrow[rr, bend right, "h_1"'] & A_2 \arrow[r, "f_2"] \arrow[rr, bend right, "h_2"'] & A_3 \arrow[r, "f_3"] \arrow[rr, bend right, "h_3"'] & A_4 \arrow[r, "f_4"] & A_5
\end{tikzcd}$$

A_i : chain complexes, f_i chain maps

① $\partial_{i+2} h_i + h_i \partial_i = f_{i+1} \circ f_i$

② $f_{i+2} h_i + h_{i+1} f_i : A_i \to A_{i+3}$ q.is

Then A_0 is q.is to the cone of A_1 and A_2

and in particular
$$\begin{tikzcd}
H(A_0) \arrow[rr, "(f_0)_*"] & & H(A_1) \arrow[dl, "(f_1)_*"] \\
& H(A_2) \arrow[ul, "\approx (f_2)_*"] &
\end{tikzcd}$$

Two double cones:

$$\left(A_0 \oplus A_1 \oplus A_2, \begin{pmatrix} \partial_0 & 0 & 0 \\ f_0 & \partial_1 & 0 \\ h_0 & f_2 & \partial_2 \end{pmatrix}\right) = (C, D)$$

$$\left(A_3 \oplus A_4 \oplus A_5, \begin{pmatrix} \partial_3 & 0 & 0 \\ f_3 & \partial_4 & 0 \\ h_3 & f_4 & \partial_5 \end{pmatrix}\right) = (C', D')$$

$$H : A_0 \oplus A_1 \oplus A_2 \longrightarrow A_3 \oplus A_4 \oplus A_5$$

$$H = \begin{pmatrix} 0 & h_1 & f_2 \\ 0 & 0 & h_2 \\ 0 & 0 & 0 \end{pmatrix}$$

consider $D'H + HD : (C, D) \to (C', D')$
it is a chain map induces 0 in homology
(H chain homotopy of this map & 0)

not a chain map

$$D'H + HD = \begin{bmatrix} h_1 f_0 + f_2 h_0 & 0 & 0 \\ h_2 h_0 & h_1 f_0 + f_2 h_0 & 0 \\ 0 & h_1 h_0 & f_3 h_1 + h_2 f_1 \end{bmatrix}$$

Framed knot (K,ℓ), $K \subseteq Y$ knot, ℓ is a homotopy class of simple closed curves (longitude) in $\partial(Y \setminus N(K))$ (which intersects the meridian ± 1 alg.) $*$ up to homotopy. This is the same as a trivialisation of normal bundle of K $N(K) \cong D^2 \times S^1$

<u>Meridian m:</u> s.c.c. in $\partial(Y \setminus N(K))$ bounding a disc in $N(K)$

$\boxed{*} \doteq m \cdot \ell = \pm 1$

If K is oriented, then <u>orient ℓ accordingly</u>

orient $m \cdot \ell = -1$ in $\partial(Y \setminus N(K))$

$Y_\ell(K)$ obtained by surgery on (K,ℓ)

$Y_\ell(K) = (Y \setminus N(K)) \cup_\varphi D^2 \times S^1$

s.t. $\partial D^2 \times \{1\}$ is mapped to ℓ.

<u>Thm</u> There exists an exact triangle

$$
\begin{tikzcd}
HF^+(Y) \arrow[rr, "f_1"] & & HF^+(Y_\ell(K)) \arrow[dl, "f_2"] \\
& HF^+(Y_{\ell+m}(K)) \arrow[ul, "f_3"] &
\end{tikzcd}
$$

Similarly for "$\wedge$" version

$Y_\ell(K) = Y' \supseteq K'$

$\ell = m'$; $\ell' = \ell + m$

$\ell' + m' = m$ (up to sign)

$(\Sigma, \underline{\alpha}, \underline{\beta}, \underline{\gamma}, \underline{\delta}, z)$ quadruple H.d.

$Y_{\alpha\beta} = Y, \quad Y_{\alpha\gamma} = Y_{\ell}(K), \quad Y_{\alpha\delta} = Y_{\ell+m}(K)$

$Y_{\beta\gamma} \cong Y_{\beta\delta} \cong Y_{\gamma\delta} \cong (S^2 \times S^1)^{\#(g-1)}$

$(\Sigma, \{\alpha_1, \dots, \alpha_g\}, \{\beta_2, \dots, \beta_g\}, z)$ describes $Y \setminus N(K)$

$\cup\!|$

$m, \ell, m+\ell$

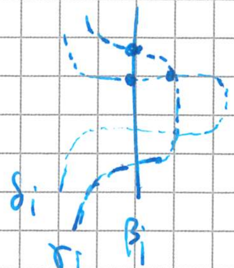
$\partial(Y \setminus N(K)) \setminus (2g-2)$ pts $\cong \Sigma \setminus (\beta_2 \cup \dots \cup \beta_g)$

$m \to \beta_1, \quad \ell \to \gamma_1, \quad m+\ell \to \delta_1$

take $\gamma_2, \dots, \gamma_g$ and $\delta_2, \dots, \delta_g$ small Hamiltonian pert. of $\beta_2, \dots, \beta_g$.

In (Σ, μ, ν, z) where $\{\mu, \nu\} \subseteq \{\beta, \gamma, \delta\}$

hat generator $\Theta_{\mu\nu}$, $\quad \theta_{\mu\nu} = [\Theta_{\mu\nu}, 0] \in CF^{\leq 0}(\Sigma, \mu, \nu)$

μ_1, ν_1 unique pt  $i > 1$ for each i

define $f_0(\xi) = F^+_{\alpha\beta\gamma}(\xi \otimes \vartheta_{\beta\gamma})$

$f_1(\xi) = F^+_{\alpha\gamma\delta}(\xi \otimes \vartheta_{\gamma\delta})$

$f_2(\xi) = F^+_{\alpha\delta\beta'}(\xi \otimes \vartheta_{\delta\beta'})$

Define $h_0(\xi) = F_{\alpha\beta\gamma\delta}(\xi \otimes \vartheta_{\alpha\beta} \otimes \vartheta_{\beta\gamma})$

$\vdots$

etc

Lemma h_0 is a chain homotopy between

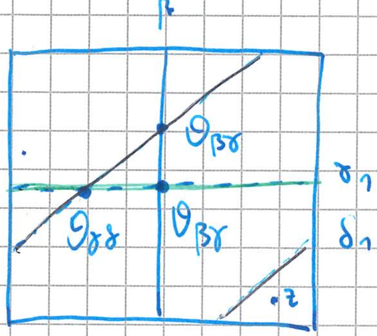
$f_1 \circ f_0$ and 0

Proof $F_{\alpha\beta\gamma\delta}(- \otimes \vartheta_{\beta\gamma} \otimes \vartheta_{\beta\delta})$ is a chain homotopy

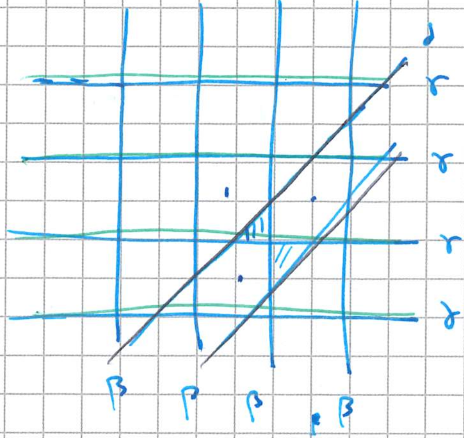
between:

$$\underbrace{F^+_{\alpha\gamma\delta}(F^+_{\alpha\beta\gamma}(- \otimes \vartheta_{\alpha\beta}) \otimes \vartheta_{\beta\gamma}}_{= f_1 \circ f_0} \quad \text{and} \quad F^+_{\alpha\beta\gamma}(\xi \otimes \overbrace{F^+_{\beta\gamma\delta}(\vartheta_{\beta\gamma} \otimes \vartheta_{\gamma\delta})}^{=0})$$

Let's see $F^{+}_{\beta\gamma\delta}(\Theta_{\beta\gamma} \otimes \Theta_{\beta\delta}) = 0$ in $\Sigma = T^2$



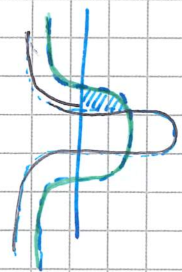
univ. cover



$k \geq 0$

Classes $\psi^{\pm}_{k}$ w $\quad n_z(\psi^{\pm}_{k}) = \frac{k(k+1)}{2}$

$\#(\mathcal{M}(\psi^{\pm}_{k})) = 1$ contributions from ψ^{+}_{k} and ψ^{-}_{k} cancel



Next ingredient in Seidel's cone detection lemma:

$h_1 f_0 + f_2 h_0$ is a q.i.

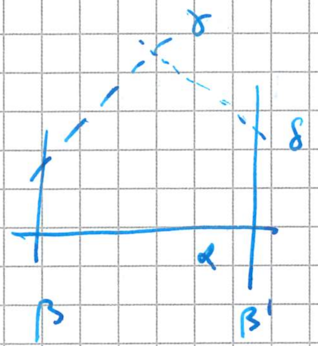
define $k: CF^{+}(\Sigma, \alpha, \beta, \gamma) \longrightarrow \Sigma(\alpha, \beta', z)$

$k(\xi) = \xi \otimes F^{+}_{\alpha\beta\gamma\delta\beta'}(\quad \Theta_{\alpha\beta} \otimes \Theta_{\beta\gamma} \otimes \Theta_{\gamma\delta} \otimes \Theta_{\delta\beta'})$

$F^+_{\alpha\beta\gamma\delta\beta'}$ counts rigid pentagons

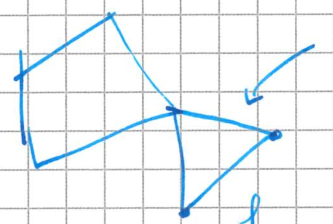
degeneration of 1-dim families

$\partial K + K\partial = f_2 \circ h_0 + h_1 \circ f_0 + \boxed{?}$

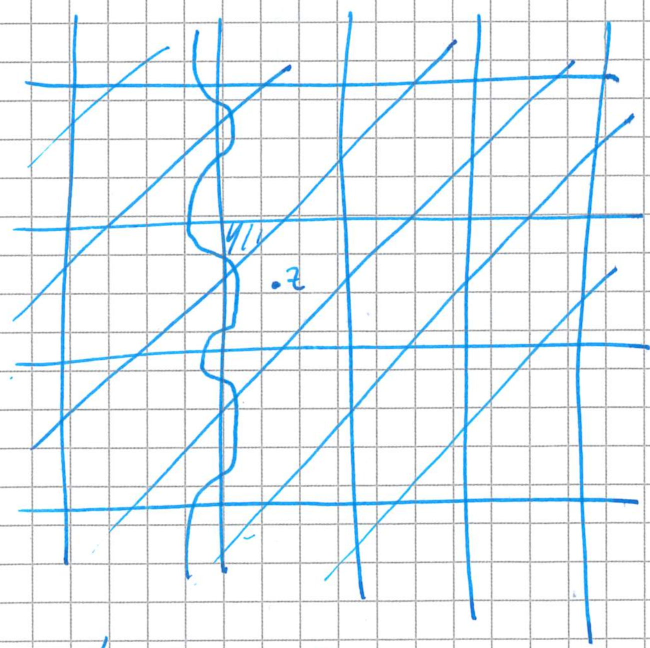


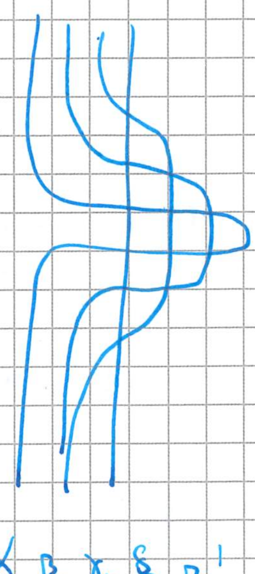
$\theta_{\beta\beta'}$
$\parallel$

$\boxed{?} = F^+_{\alpha\beta\beta'}\left(\xi \otimes F^+_{\beta\gamma\delta\beta'}\left(\theta_{\beta\gamma} \otimes \theta_{\beta\delta} \otimes \theta_{\delta\beta'}\right)\right)$



on T^2





$\alpha_1 \;\beta_1 \;\gamma_1 \;\delta_1 \;\beta_1'$

$\alpha_2 \;\beta_2 \;\gamma_2 \;\delta_2 \;\beta_2'$

Coisotropically fibred

Dehn twist along $\beta_1 \times \mathrm{Sym}^{g-1}\Sigma$