$\boxed{\text{L.X}}$

Knot Floer homology

$K \subseteq Y$ nullhomologous knot

$F \subseteq Y$ with $\partial F = K$ (Seifert surface)

on $\partial(Y \setminus N(K))$ two distinguished curves:

meridian μ bounds disc inside $N(K)$

longitude λ induced by F $\left(\lambda = F \cap \partial(Y \setminus N(K))\right)$

$(\Sigma, \underline{\alpha}, \underline{\beta}_0)$ $\quad \underline{\alpha} = (\alpha_1, \dots, \alpha_g)$

$\underline{\beta}_0 = (\beta_2, \dots, \beta_g)$

describe $Y \setminus N(K)$

identification $\partial(Y \setminus N(K)) \setminus pts$

$\cong$

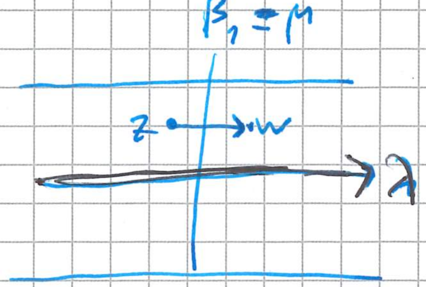
$\Sigma \setminus (\beta_2 \cup \dots \cup \beta_g)$

We orient K & λ accordingly

orient μ s.t. $\mu \cdot \lambda = -1$ in $\partial(Y \setminus N(K))$

call $\mu = \beta$ $\quad$ take $\underline{\beta} = (\beta_1, \dots, \beta_g)$

$(\Sigma, \underline{\alpha}, \underline{\beta})$ describes Y



$(\Sigma, \underline{\alpha}, \underline{\beta}, z, w)$ doubly pointed Heegaard diagram

<u>One can check:</u>

take the gradient traj. from the min to max passing through z followed by the trajectory passing through w travelled backward, we get something that is isotopic to K.

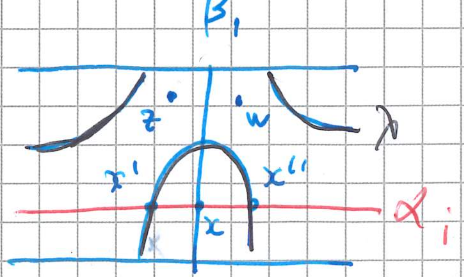
<u>Today</u> $Y_p(K) = Y_{\lambda + p\mu}(K)$

$\beta' = (\lambda, \beta_1, \dots, \beta_g)$ $\quad (\Sigma, \underline{\alpha}, \underline{\beta}')$ describes $Y_0(K)$

$H_2(Y_0(K)) \cong H_2(Y) \oplus \mathbb{Z} \leftarrow$ gen. by $[\hat{F}]$

$\hat{F} \subseteq Y_0(K)$ obtained from F + disc in sol. torus of the surgery.

$(\Sigma, \underline{\alpha}, \underline{\beta}', w)$ has a periodic domain representing $[\hat{F}]$



$x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with coordinate $x \in \alpha_i \cap \beta_i$

x', x'' closest int. pts in $\alpha_i \cap \lambda$

$x \rightsquigarrow x' \in \mathbb{T}'$ by replacing x w. x'.

Define $A(x) = \frac{1}{2} e(P) + n_{x'}(P)$

i.e. $A(x) = \frac{1}{2} \langle c_1(s(x')), [\hat{F}] \rangle$

Alexander grading.

<u>Lemma</u> $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\phi \in \pi_2(x, y)$, then

$A(x) - A(y) = n_z(\phi) - n_w(\phi)$

<u>Proof</u> $A(x) - A(y) = \frac{1}{2} \left(\langle c_1(s(x')), [\hat{F}] \rangle - \langle c_1(s(y)), [\hat{F}] \rangle \right)$

$= \varepsilon(x', y') \cdot \hat{F} = \varepsilon(x, y) \cdot \hat{F} =$ linking nr. betw. $\varepsilon(x, y)$ and $K = K \cdot \mathcal{D}(\phi) = n_z(\phi) - n_w(\phi)$

in fact $= \langle c_1(\mathfrak{s}_{y'}), \hat{F}\rangle - \langle c_1(\mathfrak{s}_{x'}), \hat{F}\rangle =$

$$\varepsilon(x', y') \cdot \hat{F}$$

(sign mistake? convention?)

$CFK^\infty(\Sigma, \underline{\alpha}, \underline{\beta}, z, w)$ generated by $[\mathbf{x}, i, j]$

$\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ $\quad i, j \in \mathbb{Z}$

$$\partial^\infty [\mathbf{x}, i, j] = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1}} \# \widehat{\mathcal{M}}(\phi) [\mathbf{y}, i - n_w(\phi), j - n_z(\phi)]$$

$$U[\mathbf{x}, i, j] = [\mathbf{x}, i-1, j-1]$$

$CFK^\infty(\Sigma, \underline{\alpha}, \underline{\beta}, z, w)$ is a filtered chain complex (filtration induced by $[\mathbf{x}, i, j] \mapsto (i, j)$)

<u>Thm</u> The filtered chain homotopy type of CFK^∞ is a topological invariant of (Y, K)

Given $m \in \mathbb{Z}$ we define $CFK^\infty(\Sigma, \underline{\alpha}, \underline{\beta}, w, m)$ as the subcomplex generated by $[\mathbf{x}, i, j]$ s.t. $A(\mathbf{x}) + i - j = m$. To see this is a subcomplex:

suppose $[y, k, l]$ appears in $\partial^\infty [x, i, j]$

$\phi \in \pi_2(x, y)$ s.t. $i - n_w(\phi) = k$

$j - n_z(\phi) = l$

$$m = A(x) + i - j = A(y) + \boxed{n_z(\phi)}$$

$A(x) - A(y) = n_z(\phi)$

$$\underbrace{-n_w(\phi) + i}_{k} \boxed{- j} = A(y) + k - l$$

$-e$

Moreover, for all $m \in \mathbb{Z}$, $CFK^\infty(\Sigma, \underline{\alpha}, \underline{\beta}, z, w)$

$\simeq CF^\infty(\Sigma, \underline{\alpha}, \underline{\beta}, w)$, $[x, i, j] \mapsto [x, i]$

j gives a filtration on $CFK^\infty(\Sigma, \underline{\alpha}, \underline{\beta}, z, w)$

$\Rightarrow$ isom induces a filtration on $CF^\infty(\Sigma, \underline{\alpha}, \underline{\beta}, w)$

On $\widehat{CF}(\Sigma, \underline{\alpha}, \underline{\beta}, w)$ we define a filtration

$$0 \subseteq \dots \subseteq F^j \widehat{CF} \subseteq F^{j+1} \widehat{CF} \subset \dots \subset \widehat{CF}$$

$F^j \widehat{CF}(\Sigma, \underline{\alpha}, \underline{\beta}, z, w)$ = subspace generated by intersections

$x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $A(x) \leq j$.

The associated graded chain complex is

$$\widehat{CFK}(\Sigma, \underline{\alpha}, \underline{\beta}, z, w) = \bigoplus_j \underbrace{\widehat{CFK}(\Sigma, \alpha, \beta, z, w, j)}_{}$$

gen. by $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$

w. $A(x) = j$

$$\partial x = \sum_{\substack{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \\ A(y) = j}} \ \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi) = 1 \\ n_z(\phi) = n_w(\phi) = 0}} \# \widehat{\mathcal{M}}(\phi)\, y$$

Call the homology $\widehat{HFK}(Y, K, j)$

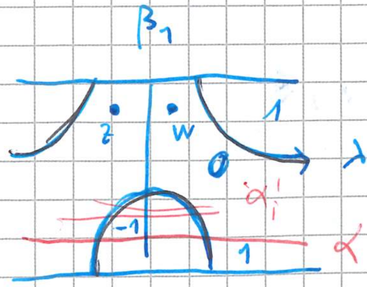
<u>Theorem</u> if $h := g(F)$ then

$$\widehat{HFK}(Y, K, j) = 0 \quad \text{if } j < -g$$

we can arrange the diagram such that

$P = \lambda - \alpha$, F is obtained by adding a disc in U_α to P along α_1

<u>Moreover</u>



$e(P) = -2h$

$\frac{1}{2} e(P) + n_{x'}(P)$

$= -h + \#$ coords x' in $\operatorname{int}(P)$

Fix $p>0$, consider $Y_{-p}(K)$ and $W_{-p}(K)$ cobordism from Y to $Y_{-p}(K)$ obtained by adding 2-handle along K.

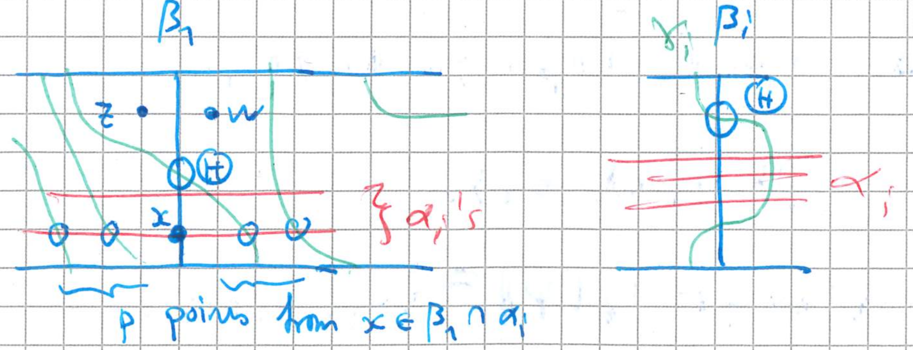
$S \subseteq W_{-p}(K)$ surface $S \cdot S = -p$ obtained by capping F with the core of the handle.

$(\Sigma, \underline{\alpha}, \underline{\beta}, \underline{\gamma})$ for $W_{-p}(K)$

$Y_{\alpha\beta} = Y$, $Y_{\alpha\gamma} = Y_{-p}(K)$, $Y_{\beta\gamma} = (S^2 \times S^1)^{\# g-1}$

$\gamma_1 = "\lambda - p\mu"$

γ_i Hamiltonian perturbation of β_i for $i > 1$.



Take $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \bar{x}^{-k}, \dots, \bar{x}^{T-k}$

replace the coord. $x \in \beta_i \cap \alpha_i$ of x with $x' \in \gamma \cap \alpha_i$

Assume $H^2(Y)=0 \Rightarrow H^2(Y_{-p}(K)) \cong \mathbb{Z}/p\mathbb{Z}$

p different Spin^c-str. in $Y_{-p}(K)$

For each $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ $x' \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ live in different Spin^c structures.

For $p \gg 0$ most Spin^c-str. will be completely $\underbrace{\text{supported in the winding region.}}_{\text{(precisely one int. in wind. region)}}$

$$\Phi : CFK^\infty(\Sigma, \alpha, \beta, z, w) \longrightarrow CF^\infty(\Sigma, \alpha, \gamma, w)$$

$$\Phi([x,i,j]) = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma} \ \sum_{\substack{\psi \in \pi_2(x,\Theta,y) \\ \mu(\psi)=0 \\ n_w(\psi)-n_z(\psi)=i-j}} \# \mathcal{M}(\psi)[y, i-n_w(\psi)]$$

<u>Magic formula</u> $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $y \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$,

$$\psi : \pi_2(x, \Theta, y)$$

then $2A(x) + 2(n_w(\psi) - n_z(\psi)) = \langle c_1(s_1(\psi)), [S] \rangle - p$

We define $I(p,m) \subseteq \mathrm{Spin}^c(Y_{-p}(K))$ s.t.

$s \in I(p,m) \Leftrightarrow s = r|_{Y_{-p}(K)}$, $r \in \mathrm{Spin}^c(W_{-p}(K))$

with $\langle c_1(r), S \rangle + p \equiv 2m \pmod{2p}$

$$CF^\infty(\Sigma, \alpha, \gamma, w, [m]) = \bigoplus_{s \in I(p,m)} CF^\infty(\Sigma, \alpha, \gamma, w, s)$$

Φ maps $CFK^\infty(\Sigma, \underline{\alpha}, \beta, z, w, m)$ to

$$CF^\infty(\Sigma, \alpha, \gamma, w, [m])$$

We restrict attention to m s.t. $CF^\infty(\Sigma, \alpha, \gamma, w, [m])$

is supported in the winding region $(*)$

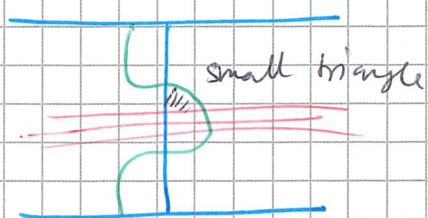
$$\Phi = \underbrace{\Phi_0}_{\text{small area triangles}} + \underbrace{\ell}_{\text{counts large area triangles}}$$

There is a small area triangle in

$\pi_2(x, \Theta, y)$ iff $y = \bar{x}^{(i)}$

In that case there is a unique ψ_0

Then $\Phi_0([x, i, j]) = [\bar{x}^{(i)}, \ ]$



For m such that $(*)$ holds

Φ_0 is <u>an iso.</u>