L. XI

Erratum We used $J_t = \mathrm{Sym}^g(j_t)$ on $\mathrm{Sym}^g \Sigma$

j_t path of complex structures on Σ

This does not make sense :(

Smooth structure on $\mathrm{Sym}^g \Sigma$ depends on j_t.

So J_t is a path of cplx structures, but for different smooth structures on $\mathrm{Sym}^g \Sigma$

Solution: For defining HF, go back to O-S3.

Fix j on Σ and take J_t to be a perturbation of $\mathrm{Sym}^g j$

However, we run into problems:

1. surely in showing isomorphism with the cylindrical reformulation
2. maybe in showing that HF does not depend on j

All smooth structures in $\mathrm{Sym}^g \Sigma$ constructed this way are diffeomorphic, so the problem is probably not so serious.

Bordered Floer homology

$Y = Y_0 \cup_{\Sigma} Y_1$ with $\partial Y_0 = -\partial Y_1 = \Sigma = F$

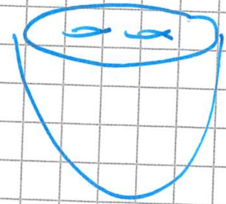
we want to compute $\widehat{HF}(Y)$ from the pieces

To do this we will define:

- a dg-algebra $\mathcal{A}$ associated to Σ (decorated with some extra structure)
- an A_∞-module over $\mathcal{A}$ $\widehat{CFA}(Y_0)$
- a dg-module over $\mathcal{A}$ $\widehat{CFD}(Y_1)$

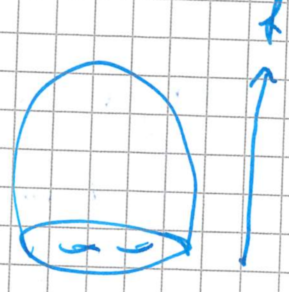
such that $\widehat{CF}(Y) \simeq \widehat{CFA}(Y_0) \otimes^L \widehat{CFD}(Y_1)$

Y 3-mfd w. $\partial Y = F$

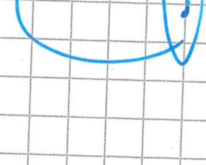


consider (f, r) such that

$f : Y \to \mathbb{R}$ Morse function w. one max and one min

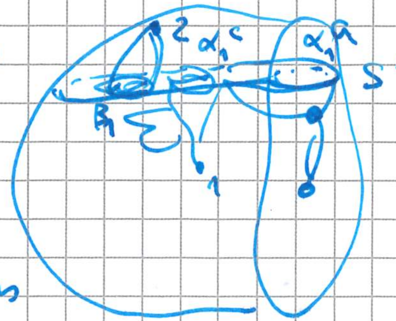


- $f|_{\partial Y}$ Morse function w. one max & one min.
- the max/min of $f|_{\partial Y}$ coincide w. the max min of f



- index 1 crit pts of $f|_{\partial Y}$ are index 1 crit pts of f.
- v pseudogradient of f which is tangent to ∂Y.
- f should be self-indexing

Then $\overline{\Sigma} = f^{-1}(\frac{3}{2})$, $\partial \overline{\Sigma} \cong S^1$



$g = g(\overline{\Sigma})$, $\beta_1, \dots, \beta_g$ intersections between $\overline{\Sigma}$ and the stable manifolds of the index 2 crit pts. of f

$\alpha_1^c, \dots, \alpha_{g-k}^c$ $\quad k = g(\partial Y)$

intersections between the unstable manifolds of the interior index 1 crit pts and $\overline{\Sigma}$.

$\alpha_1^a, \dots, a_{2k}^a$ <u>arcs</u> intersections between unstable mflds of index 1 crit pts of f and ∂Y an $\overline{\Sigma}$.

Call $\underline{\beta} = \{\beta_1, \dots, \beta_g\}$, $\underline{\alpha}^c = \{\alpha_1^c, \dots, \alpha_{g-k}^c\}$

$$\underline{\alpha}^a = \{\alpha_1^a, \dots, \alpha_{2k}^a\}, \quad \underline{\alpha} = \underline{\alpha}^c \cup \underline{\alpha}^a$$

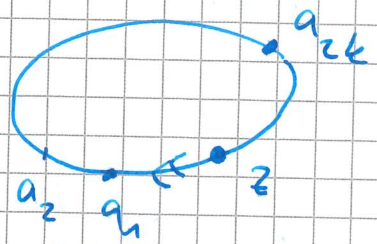
$\underline{a} = \underline{\alpha} \cap \partial\Sigma$. Choose a basepoint

$z \in \partial\Sigma \setminus \underline{a}$ order $\underline{a} = \{a_1, \dots, a_{2k}\}$ according to orientation

$M : \underline{a} \longrightarrow \{1, \dots, 2k\}$ matching

$M(a) = i$ iff a is an endpoint of α_i^a

for $a \in \underline{a}$



$(\bar{\Sigma}, \bar{\underline{\alpha}}, \beta, z)$ pointed bordered Heegaard diagram

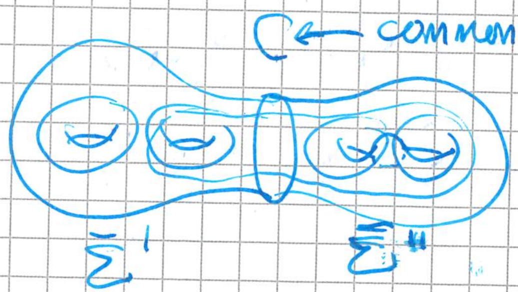
$(\partial\Sigma, \underline{a}, M, z)$ pointed matched circle

Suppose that $(\bar{\Sigma}', \bar{\underline{\alpha}}', \beta', z') \rightarrow Y_0$ and $(\bar{\Sigma}'', \bar{\underline{\alpha}}'', \beta'', z'') \rightarrow Y_1$

bordered H. d. s.t. they induce the same pointed matched circle up to inversing orientation.

You can glue them to obtain a Heegaard diagram

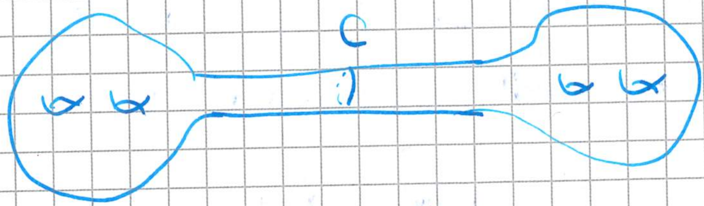
$(\Sigma, \underline{\alpha}, \beta, z)$ for a closed 3-mfd $Y = Y_0 \cup_\partial Y_1$



$\widehat{CF}(Y)$ generated by tuples of int. points

differential counts holom. curves in $\mathbb{R}\times[0,1]\times\Sigma$

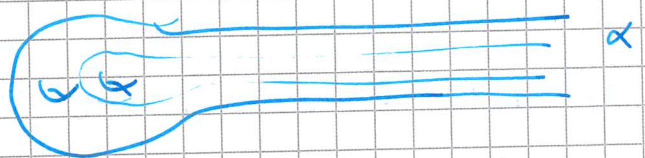
$\to$ insert a long neck around C.



In the limit: $\Sigma' \sqcup \Sigma''$

obtained by completing $\bar{\Sigma}'$, $\bar{\Sigma}''$ by adding

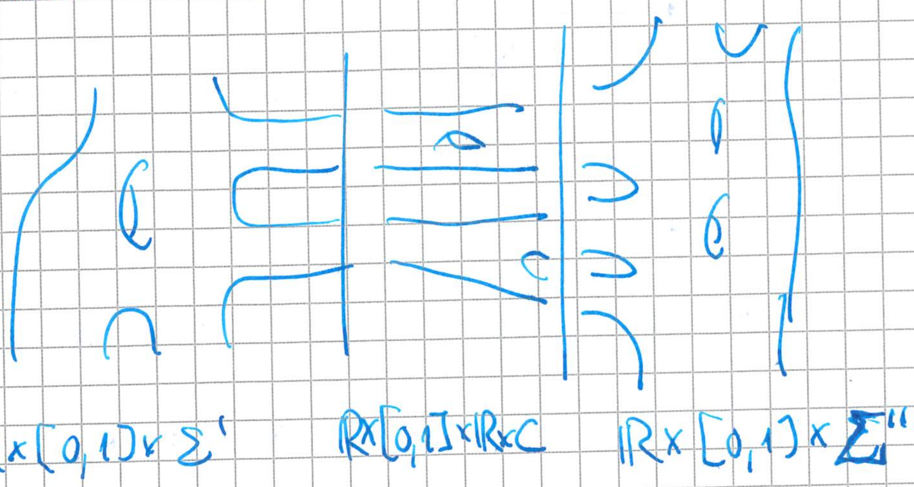
half a cylinder to $\partial\bar{\Sigma}'$, $\partial\bar{\Sigma}''$



Holomorphic curves in $\mathbb{R}\times[0,1]\times\Sigma$

converge to buildings w. both vertical and

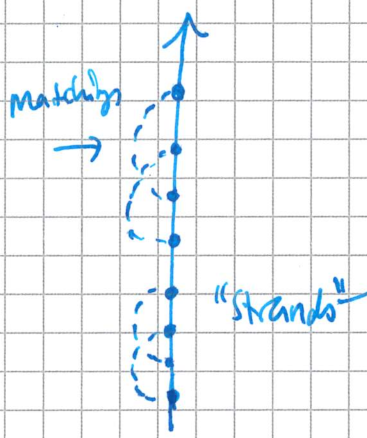
horizontal level



Algebra associated to a pointed matched circle

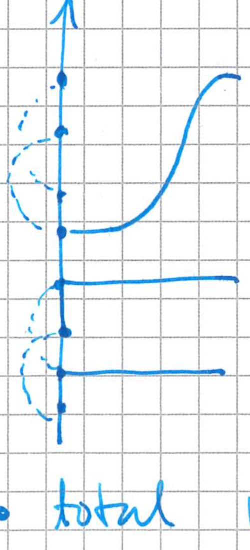
$\mathcal{Z} = (C, \underline{a}, M, z)$ $\qquad C \setminus \{z\} \cong \mathbb{R} \rightsquigarrow A(\mathcal{Z})$

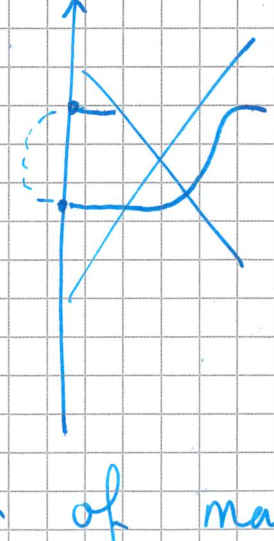
algebra over $\mathbb{F}_2$



as a vector space: generated by "strand diagrams" collection of lines in $[0,1] \times \mathbb{R}$ with endpoints on $\{0\} \times \underline{a}$ and $\{1\} \times \underline{a}$ up to homotopy fixing ∂

- each strand is either horizontal or pointing upward
- horizontal strands are in matched pairs
- starting points of nonhorizontal strands are not matched to any other starting point.



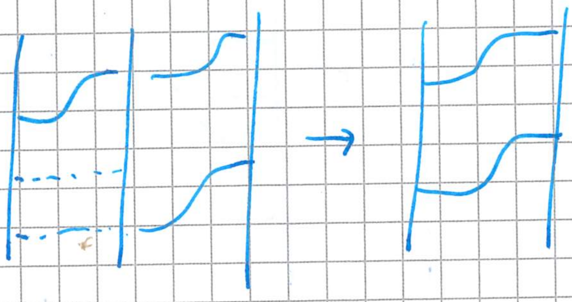


- total number of matched pairs hit by the starting/endpoints of some strand is k.

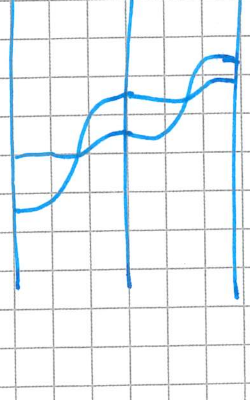
Multiplication is defined by concatenation subject to the following rules:

① two strands can be concatenated if the endpoints of the 1st is equal to the starting point of the 2nd.

② if a horizontal strand is concatenated to nonhoriz. strand, the matched horizontal strand is deleted

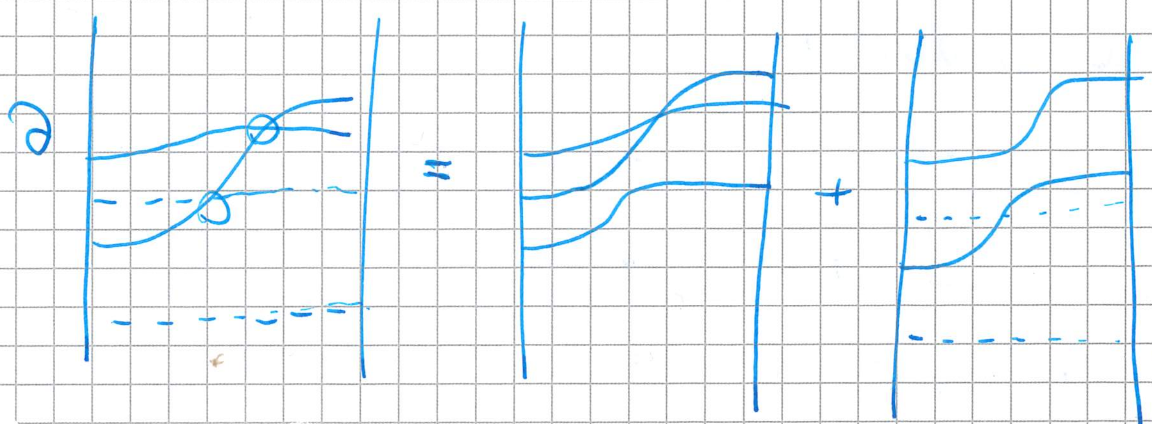


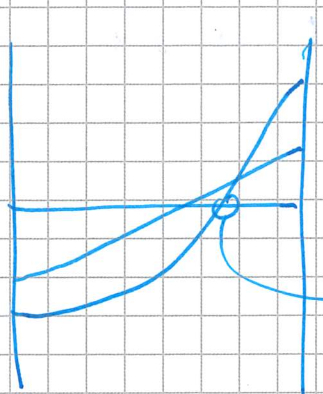
③ If after concatenating a double crossing is created, the product is set to 0


$= 0$

The differential is a sum of ways to resolve crossings, according to the following rules:

1. if a crossing with a horizontal line is resolved, the matched horizontal line is deleted
2. if a double crossing is created, that term is set to zero





resolving this
creates a double crossing.

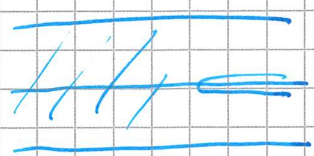
$\overbrace{\text{Reeb chords on } S^1}$

upwards strands are oriented arcs in $\mathbb{C}$

connecting points $\underline{a}$ and not passing through z.

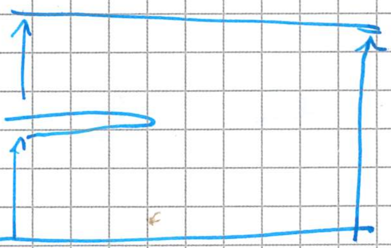
← Leg. subarcs

the product encodes holomorphic curves in

$\mathbb{R}\times\mathbb{C}$ of the form 

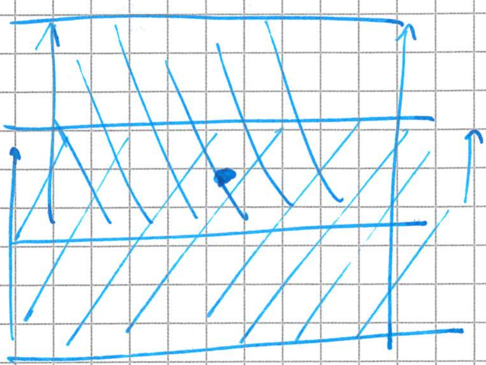
split curve

Differential counts curves of following two types



join curve

(associated w. resolving a crossing with a horizontal strand)



an odd shuttle curve

associated to resolving a crossing between two upward strands.

$S, T \subseteq \{1, \dots, 2k\}$, $\mathcal{A}_{S,T}(z)$ generated by strand diagrams w. strands starting from points a with $M(a) \in S$, ending at pts a' w. $M(a') \in T$

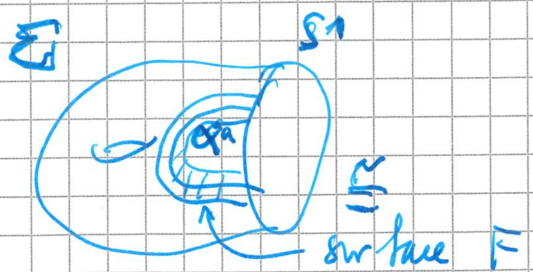
$$\mathcal{A}(z) = \bigoplus_{S,T} \mathcal{A}_{S,T}(z) \quad \text{subcomplexes}$$

$$S \rightsquigarrow \mathbb{D}_S = \alpha_{i_1}^a \times \dots \times \alpha_{i_k}^a \qquad S = \{i_1, \dots, i_k\}$$

$\subseteq \mathrm{Sym}^k(F)$

surface defined by matching

Lagrangian plane



$$\mathbb{D}_S^+ = (\alpha_{i_1}^a)^+ \times \dots \times (\alpha_{i_k}^a)^+$$

obtained by pushing the ends of $\alpha_{i_j}^a$ towards z (w. <u>pos</u> Reeb flow)

Thm (Auroux) $\mathcal{A}_{S,T} \cong CF^*(D^r_S, D_T)$

product in $\mathcal{A}(Z)$ comes from the triangle product and high. order operations vanish.